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Soft Γ-Semirings

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Abstract. In this paper, the definitions of soft Γ-semirings and soft sub-Γ-semi rings are introduced with the aid of the concept of soft set theory introduced by Molodtsov. Then, some of their properties and structural characteristics are investigated and discussed. Moreover, the definition of soft Γ-homomorphism and soft Γ-isomorphism are given and construct first, second and third isomorphism theorems of soft Γ-semiring, respectively.

Keywords. Soft Sets, Fuzzy Sets, Semirings, Γ-semiring.

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1. Introduction

Uncertain data modelling was investigated by many researchers in economics, engineering, environmental sciences, sociology, medical science and many other fields. The process in classical mathematics may not be competent owing to the fact that the assorted uncertainties deriving in these fields. In this context, mathematical theories such as probability theory, fuzzy set theory [1], rough set theory [2] were established by researchers to modelling uncertainties arising in the stated fields. In 1999, Molodtsov [3] made a new viewpoint of substantial theoretical approaches: the concept of soft set theory which is more convenient than classical ideologies and can be seen as a outstanding mathematical tool relates with uncertainties. After Molodtsov's work, some different applications of soft sets were studied in [4, 5, 6, 7].

The algebraic structure of soft set theories has been studied progressively in recent years. Aktaş and Çağman [8] investigated basic properties of soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups, and derived some related properties. Furthermore, Maji et al. [9, 10] presented the definition of fuzzy soft set. The concept of fuzzy soft groups which is a generalization of soft groups were given in [11] and [12]. In 2010, a tentative approach between fuzzy sets (rough sets) and soft sets were studied by Feng et al. in [13].

On the other hand soft rings, soft ideals on soft rings and idealistic soft rings were defined in [14]. After these studies the notion of fuzzy soft rings and fuzzy soft ideals were discussed in [15]. In addition to this in [16] the concept of soft BCH-algebra was introduced and some of their properties and structural characteristics were mentioned. The notion of soft semirings are investigated in [17] which is useful for dealing with problems in different areas of applied mathematics and information sciences. The semiring structure provides an algebraic framework for modelling and investigating the key factors in these problems. Then, N. Nobusawa [18] introduced the notion of Γ-ring, as more general than ring. After that, the weakened conditions of the definition of Γ-ring were studied in [19]. The generalization of Γ-ring and Γ-semiring were

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introduced by [20]. In [21] Jun and Lee studied the concept of fuzzy Γ-ring and [22] defined the soft Γ-rings and idealistic soft Γ-rings with their basic properties. The extension of Γ-semiring to quasi ideals were identified by [23, 24, 25] with incompatible style.

In this paper, we introduce the concept of soft Γ-semiring which extends the notion of soft Γ-ring theory and deal with some of its algebraic properties by giving several examples.

2. Preliminaries

Definition 1. A pair (ρ, W) is called a soft set over V, where ρ is a mapping from W to $P(V)$ [3].

Definition 2. Let $(\rho, W), (\sigma, Y)$ be soft sets over a common universe V.

i) If $W \subseteq Y$ and $\rho(\omega) \subseteq \sigma(\omega)$ for all $\omega \in W$ then we say that (ρ, W) is a soft subset of (σ, Y),denoted by $(\rho, W) \widetilde{\subseteq} (\sigma, Y)$.

ii) If (ρ, W) is a soft subset of (σ, Y) and (σ, Y) is a soft subset of (ρ, W), then we say that (ρ, W) is soft equal to (σ, Y), denoted by $(\rho, W) = (\sigma, Y)$.

Definition 3. i) Let (ρ, W) and (σ, Y) be two soft set over a common universe V.

$$(\psi, Z) = (\rho, W) \widetilde{\cap}_{\Re} (\sigma, Y)$$

is said to be restricted-intersection of (ρ, W) and (σ, Y), where (ψ, Z) is soft set, $Z = W \cap Y \neq \emptyset$ and the mapping ψ is defined by

$$\begin{array}{llll} \psi: & Z & \rightarrow & P(V) \\ & z & \rightarrow & \psi(z) = \rho(z) \cap \sigma(z). \end{array}$$

ii) Let $\{(\rho_i, W_i) : i \in I\}$ be non-empty family soft sets. The restricted-intersection of a non-empty family soft sets is defined by

$$(\psi, Y) = \left(\widetilde{\cap}_{\Re}\right)_{i \in I} (\rho_i, W_i),$$

where (ψ, Y) is a soft set, $Y = \bigcap_{i \in I} W_i \neq \emptyset$ and $\psi(y) = \bigcap_{i \in I} \rho_i(y)$ for every $y \in Y$ [17].

Definition 4. i) Let (ρ, W) and (σ, Y) be two soft set over a common universe V.

$$(\psi, Z) = (\rho, W) \widetilde{\cap}_{\mathscr{E}} (\sigma, Y)$$

is called extended-intersection of (ρ, W) and (σ, Y), where (ψ, Z) is soft set and (ψ, Z) satisfying the following conditions:

- $Z = W \cup Y$
- $\psi(z) = \begin{cases} \rho(z) & , if\ z \in W \backslash Y \\ \sigma(z) & , if\ z \in Y \backslash W \\ \rho(z) \cap \sigma(z) & , if\ z \in c \in W \cap Y. \end{cases}$

ii) Let $\{(\rho_i, W_i) : i \in I\}$ be non-empty family soft sets. The extended-intersection of a non-empty family soft sets is defined by

$$(\psi, Y) = \left(\widetilde{\cap}_{\mathscr{E}}\right)_{i \in I} (\rho_i, W_i),$$

where (ψ, Y) is a soft set, $Y = \bigcup_{i \in I} W_i, \psi(y) = \bigcap_{i \in I} \rho_i(y)$ and $I(y) = \{i : i \in W_i\}$ for every $y \in Y$ [7],[17] .

Definition 5. i) Let (ρ,W) and (σ,Y) be two soft set over a common universe V.

$$(\psi,Z)=(\rho,W)\widetilde{\cup}_{\mathfrak{R}}(\sigma,Y)$$

is said to be restricted union of (ρ,W) and (σ,Y), where (ψ,Z) is soft set, $Z=W\cap Y\neq\emptyset$, and the mapping ψ is defined by

$$\begin{array}{rccl}\psi: & Z & \to & P(V)\\ & z & \to & \psi(z)=\rho(z)\cup\sigma(z)\end{array}$$

[7].

ii) Let $\{(\rho_i,W_i):i\in I\}$ be non-empty family soft sets. The restricted-union of a non-empty family soft sets is defined by

$$(\psi,Y)=\left(\widetilde{\cup}_{\mathfrak{R}}\right)_{i\in I}(\rho_i,W_i),$$

where (ψ,Y) is a soft set, $Y=\bigcap_{i\in I}W_i\neq\emptyset$ and $\psi(y)=\bigcup_{i\in I}\rho_i(y)$ for every $y\in Y$ [16].

Definition 6. i) Let (ρ,W) and (σ,Y) be two soft set over a common universe V.

$$(\psi,Z)=(\rho,W)\widetilde{\cup}_{\mathscr{E}}(\sigma,Y)$$

is said to be extended union of (ρ,W) and (σ,Y), where (ψ,Z) is a soft set, $Z=W\cup Y$, and the mapping ψ is defined by

$$\psi(z)=\begin{cases}\rho(z) & ,if\ z\in W\backslash Y\\ \sigma(z) & ,if\ z\in Y\backslash W\\ \rho(z)\cup\sigma(z) & ,if\ z\in c\in W\cap Y.\end{cases}$$

ii) Let $\{(\rho_i,W_i):i\in I\}$ be non-empty family soft sets. The extended-union of a non-empty family soft sets is defined by

$$(\psi,Y)=\left(\widetilde{\cup}_{\mathscr{E}}\right)_{i\in I}(\rho_i,W_i),$$

where (ψ,Y) is a soft set, $Y=\bigcup_{i\in I}W_i$, $\psi(y)=\bigcup_{i\in I}\rho_i(y)$ and $I(y)=\{i:i\in W_i\}$ for every $y\in Y$ [7].

Definition 7. i) Let (ρ,W) and (σ,Y) be two soft set over a common universe V.

$$(\psi,Z)=(\rho,W)\widetilde{\Lambda}(\sigma,Y)$$

is called Λ-intersection of (ρ,W) and (σ,Y), where (ψ,Z) is soft set, $Z=W\times Y$ and $\psi(w,y)=\rho(\omega)\cap\sigma(y)$ for every $(\omega,y)\in W\times Y$.

ii) Let $\{(\rho_i,W_i):i\in I\}$ be non-empty family soft sets. The Λ-intersection of a non-empty family soft sets is defined by

$$(\psi,Y)=\widetilde{\Lambda}_{i\in I}(\rho_i,W_i),$$

where (ψ,Y) is a soft set, $Y=\Pi_{i\in I}W_i$ and $\psi(y)=\bigcap_{i\in I}\rho_i(y)$ for every $y=(y_i)_{i\in I}\in Y$ [6, 17].

Definition 8. i) Let (ρ,W) and (σ,Y) be two soft set over a common universe V.

$$(\psi,Z)=(\rho,W)\widetilde{\vee}(\sigma,Y)$$

is called $\vee$-union of (ρ,W) and (σ,Y), where (ψ,Z) is soft set, $Z=W\times Y$ and $\psi(w,y)=\rho(\omega)\cup\sigma(y)$ for every $(\omega,y)\in W\times Y$.

ii) Let $\{(\rho_i, W_i : i \in I)\}$ be non-empty family soft sets. The $\vee$-union of a non-empty family soft sets is defined by

$$(\psi, Y) = \widetilde{\vee}_{i\in I}(\rho_i, W_i),$$

where (ψ, Y) is a soft set, $Y = \Pi_{i\in I} W_i$ and $\psi(y) = \bigcup_{i\in I} \rho_i(y)$ for every $y = (y_i)_{i\in I} \in Y$ [6, 17].

Definition 9. i) Let (ρ, W) and (σ, Y) be two soft sets over a common universe V_1 and V_2 respectively. The cartesian product of two soft sets (ρ, W) and (σ, Y) is defined by

$$(Z, W \times Y) = (\rho, W) \times (\sigma, Y),$$

where $(Z, W \times Y)$ is a soft set, and $\psi(\omega, y) = \rho(\omega) \times \sigma(y)$ for every $(w, y) \in W \times Y$[6].

ii) Let $\{(\rho_i, W_i) : i \in I\}$ be non-empty family soft sets over $V_i, i \in I$. The cartesian product of a non-empty family soft sets $\{(\rho_i, W_i) : i \in I\}$ over the universes V_i, is defined by

$$(\psi, Y) = \widetilde{\Pi}_{i\in I}(\rho_i, W_i),$$

where (ψ, Y) is a soft set, $Y = \Pi_{i\in I} W_i$ and $\psi(y) = \Pi_{i\in I} \rho_i(y)$ for all $y = (y_i)_{i\in I} \in Y$ [16].

Definition 10. i) Let (ρ, W) be soft set over a common universe V. Then (ρ, W) is said to be a relative null soft set, denoted by N_W, if $\rho(e) = \emptyset$ for every $e \in W$.

ii) (ρ, W) is said to be relative whole soft, denoted by $\mathscr{W}_W$, if $\rho(e) = V$ for every $e \in W$ [16].

Definition 11. Let (ρ, W) and (σ, Y) be two softs set over a common universe V_1 and V_2, respectively, and $f : V_1 \to V_2, g : W \to Y$ be two functions. (f, g) is said to be a soft function from (ρ, W) to (σ, Y), denoted by $(f, g) : (\rho, W) \to (\sigma, Y)$ if the following condition

$$f(\rho(\omega)) = \sigma(g(\omega))$$

satisfies for all $w \in W$. If f and g are injective (resp. surjective, bijective), then we say that (f, g) is injective(resp. surjective, bijective)[16].

Lemma 1. Let $(\rho, W), (\sigma, Y)$ and (ψ, Z) be soft sets over V_1, V_2 and V_3, respectively. If

$$(f, g) : (\rho, W) \to (\sigma, Y)$$

and

$$\left(f', g'\right) : (\sigma, Y) \to (\psi, Z)$$

are two soft functions, then

$$\left(f' \circ f, g' \circ g\right) : (\rho, W) \to (\psi, Z)$$

is a soft function.

Definition 12. Let (ρ, W) and (σ, Y) be two soft sets over V_1 and V_2, respectively, (f, g) is a soft function such as

$$(f, g) : (\rho, W) \to (\sigma, Y).$$

The image of (ρ, W) is soft set over V_1 defined by

$$f(\rho)(y) = \begin{cases} V_{1_{g(\omega)=y}} f(\rho(\omega)) & if y \in Img \\ \emptyset & otherwise \end{cases}$$

for all $y \in Y$. The pre-image of (σ, Y) is soft set over V_1 defined by $f^{-1}(\sigma)(\omega) = f^{-1}(\sigma(\rho(\omega)))$ for all $\omega \in W$.

It is clear that $(f,g)(\rho,W)$ and $f^{-1}(\sigma)(\omega)$ are soft subsets of (σ,Y) and $(f,g)^{-1}(\sigma,Y)$, respectively. If ρ is the identity function on W, the soft sets $(f(\rho),W)$ and (ρ,W) are as given in [8] and [16].

Definition 13. Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S\times\Gamma\times S$ (images to be denoted by $a\alpha b$ for all $a,b\in S$ and $\alpha\in\Gamma$) satisfying the following conditions

i) $(a+b)\alpha c = a\alpha c + b\alpha c$
ii) $a\alpha(b+c) = a\alpha b + a\alpha c$
iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$
iv) $a\alpha(b\beta c)b = (a\alpha b)\beta c$ for all $a,b,c\in S$ and for all $\alpha,\beta\in\Gamma$ [24].

Example 2.1. Let $\mathbb{Q}$ be set of rational numbers. $(S,+)$ be the commutative semigroup of all 2×3 matrices over $\mathbb{Q}$ and $(\Gamma,+)$ be commutative semigroup of all 3×2 matrices over $\mathbb{Q}$. Define $W\alpha Y$ usual matrix product of W,α and Y ; for all $W,Y\in S$ and for all $\alpha\in\Gamma$. Then S is a Γ-semiring but not a semiring [24].

Remark 2.1. Let $\mathbb{N}$ be the set of natural numbers and $\Gamma=\{1,2,3\}$. Define the mapping $\mathbb{N}\times\Gamma\times\mathbb{N}\to\mathbb{N}$ by $a\alpha b = a\cdot\alpha\cdot b$ (usual product of a,α and b); for all $a,b\in\mathbb{N},\alpha\in\Gamma$. Then $\mathbb{N}$ is a Γ-semiring given in [25]. But Γ is not an additive semigroup, hence it is not a Γ-semiring according to [23].

Example 2.2. Let $\mathbb{N}$ be the set of natural numbers and $\Gamma=\{1,2,3\}$ $(\mathbb{N},max)$ and (Γ,max) are commutative semigroups. Define the mapping $\mathbb{N}\times\Gamma\times\mathbb{N}\to\mathbb{N}$, by $a\alpha b = min\{a,\alpha,b\}$, for all $a,b\in\mathbb{N},\alpha\in\Gamma$. Then $\mathbb{N}$ is a Γ-semiring [24].

Example 2.3. Let $\mathbb{Q}$ be set of rational numbers and $\Gamma=\mathbb{N}$ be the set of natural numbers $(\mathbb{Q},+)$ and $(\mathbb{N},+)$ are commutative semigroups. Define the mapping $\mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\to\mathbb{Q}$ by $a\alpha b$ usual product of $a,\alpha,b;a,b\in\mathbb{Q},\alpha\in\Gamma$. Then $\mathbb{Q}$ is a Γ-semiring [24].

3. Soft Γ-Semiring

Let $S\neq\varnothing$ be a Γ-semiring. R will allude to any triplet relation the midst of a component of S and a component of Γ and a component of S, that is, esoterically R is a subset of $S\times\Gamma\times S$. In this way, a set valued function $\psi : N\to P(S)$ can be defined as

$$\psi(y) = \{s\in S : R(y,\alpha,s), \forall\alpha\in\Gamma\}$$

for all $y\in N$. The pair (ψ,N) is then a soft set over S, which produced from the relation R. The set

$$Supp(\psi,N) = \{y\in N : \psi(y)\neq\emptyset\}$$

is called a support of the soft set (ψ,N). The soft set (ψ,N) non-null if $Supp(\psi,N)\neq\emptyset$ [16, 17].

Definition 14. A non-empty subset T of S is said to be a sub-Γ- semiring of S if $(T,+)$ is a subsemigroup of $(S,+)$ and $a\alpha b\in T$; for all $a,b\in T$ and for all $\alpha\in\Gamma$ [24].

Definition 15. Let (ψ,N) be a non-null soft set over a Γ-semiring S. Then (ψ,N) is called a soft Γ-semiring over S if $\psi(x)$ is a sub-Γ-semiring of S for all $y\in Supp(\psi,N)$. This denoted by (ψ,N).

Example 3.1. For consider the additively abelian groups $\mathbb{Z}_8=\{0,1,2,3,4,5,6,7\}$ and $\Gamma=\{0,2,4,6\}$. Let $\cdot:\mathbb{Z}_8\times\Gamma\times\mathbb{Z}_8\to\mathbb{Z}_8, (y,\alpha,s)=y\alpha s$. $\mathbb{Z}_8$ is a Γ-semiring.

Let $N=\mathbb{Z}_8$ and $\psi : N\to P(\mathbb{Z}_8)$ be a set valued function defined by

$$\psi(y) = \{s\in\mathbb{Z}_8 : R(y,\alpha,s)\leftrightarrow, (y,\alpha,s)\in\{0,4,6\}, \forall\alpha\in\Gamma\}$$

for all $y \in N = \mathbb{Z}_8$. Then

$$\psi(0) = \psi(2) = \psi(4) = \psi(6) = \mathbb{Z}_8$$

$$\psi(1) = \psi(3) = \psi(5) = \psi(7) = \{0,2,4,6\}$$

are sub-Γ-semirings of $\mathbb{Z}_8$. Hence (ψ,N) is a soft-Γ-semiring over $\mathbb{Z}_8$.

Example 3.2. A soft Γ-semiring (ψ,N) be a non-null for which N is a singleton a sub-Γ-semiring. Thus sub-Γ-semirings and classical Γ-semirings are specific type of soft Γ-semirings.

Example 3.3. Let

$$S = \{[\bar{0}\ \bar{0}], [\bar{1}\ \bar{0}], [\bar{0}\ \bar{1}], [\bar{1}\ \bar{1}]\} \subseteq (\mathbb{Z}_2)_{1\times 2}.$$

and

$$\Gamma = \left\{ \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} \\ \bar{0} \end{bmatrix} \right\} \subseteq (\mathbb{Z}_2)_{1\times 2}$$

are additively abelian groups with matrices addition. It can be easily checked that S is a Γ-semiring defined as

$$\begin{aligned} \cdot : R \times \Gamma \times R &\longrightarrow R \\ (x,\alpha,y) &\longrightarrow x\alpha y. \end{aligned}$$

Take $N = \{[\bar{0}\ \bar{0}], [\bar{1}\ \bar{1}]\} \subseteq S$ and $\varphi : N \longrightarrow P(S)$ be a set valued function defined by $\varphi = \{y \in S \,|\, R(x,\alpha,y) \Leftrightarrow x\alpha y \in \{[\bar{0}\ \bar{0}], [\bar{1}\ \bar{1}]\}\}$. Hence (φ,N) is a soft Γ-semiring over S.

Example 3.4. Let $S = \mathbb{Z}_0^-$ be set off all non-positive integer and $\Gamma = a,b$. Then with the addition defined as follows:

+	a	b
a	a	b
b	b	a

Now we can easily see that Γ is commutative. If we define

$$\begin{aligned} S \times \Gamma \times S &\longrightarrow S \\ (x,a,y) &\longrightarrow 0 \\ (x,b,y) &\longrightarrow -(xy) \end{aligned}$$

then S forms a Γ-semiring. $N = \mathbb{Z}_0^- \setminus \{-2\}$. Then $\psi = N \longrightarrow P(S)$, (ψ,N) is a soft Γ-semiring.

Proposition 1. Let (ρ,W) and (σ,W) be soft Γ semirings over Γ-semiring S. The restricted intersection $(\rho,W)\,\widetilde{\cap}_{\Re}\,(\sigma,W)$ is a soft Γsemiring over S if it is non-null.

Proof. By Definition 2.3 (i), we have that $(\rho,W)\,\widetilde{\cap}_{\Re}\,(\sigma,W) = (\psi,W)$, where $\psi(\omega) = \rho(\omega) \cap \sigma(\omega)$ for all $\omega \in W$. We assume that (ψ,W) is a non-null soft set over S. If $\omega \in Supp(\psi,W)$, then $\psi(\omega) = \rho(\omega) \cap \sigma(\omega) \neq \emptyset$. We know that (ρ,W) and (σ,W) are both soft Γ semirings over S, and so, the nonempty sets $\rho(\omega)$ and $\sigma(\omega)$ are both sub-Γ semiring of S (From definition 3.2). Thus, $\psi(w)$ is a sub-Γ-semiring of S for all $\omega \in Supp(\psi,W)$. In this position, $(\psi,W) = (\rho,W)\,\widetilde{\cap}_{\Re}\,(\sigma,W)$ is a soft Γ semiring over S ■

Corollary 1. Let $\{(\rho_i,W) : i \in I\}$ be a nonempty family of soft Γ-semiring over S. Then the restricted intersection $\left(\widetilde{\cap}_{\Re}\right)_{i\in I}(\rho_i,W)$ is a soft Γ-semiring over S if it is non-null.

Proof. Straight forward ■

Theorem 1. Let $(\rho_i, W_i)_{i\in I}$ be a nonempty family of soft-Γ-semirings over S. Then the restricted intersection $\left(\widetilde{\cap}_{\Re}\right)_{i\in I}(\rho_i, W_i)$ is a soft Γ-semiring over S if it is non-null.

Proof. From definition 2.3(ii), we have that $\left(\widetilde{\cap}_{\Re}\right)_{i\in I}(\rho_i, W_i) = (\psi, Y)$, where $Y = \bigcap_{i\in I} W_i \neq \emptyset$, and $\psi(y) = \bigcap_{i\in I}\rho_i(y)$ for every $y \in Y$.

We assume that (ψ, Y) is non-null. Let $y \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$ and so we have $\rho_i(y) \neq \emptyset$ for every $i \in I$. From the hypothesis, we know that $\{(\rho_i, W_i) : i \in I\}$ is a nonempty family of soft-Γ-semiring over S, by definition 3.2 $\rho_i(y)$ is a sub-Γ-semiring of S, that is, $\psi(y)$ is a sub-Γ-semiring of S for all $y \in Supp(\psi, Y)$ and so (ψ, Y) is a soft Γ semiring over S. ■

Theorem 2. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semiring over S. Then the extended intersection $\left(\widetilde{\cap}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i)$ is a soft Γ-semirings over S.

Proof. From definition 2.4 (ii), we have that $\left(\widetilde{\cap}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i) = (\psi, Y)$, where $Y = \bigcup_{i\in I} W_i$, and $\psi(y) = \bigcap_{i\in I}\rho_i(y)$ for all $y \in Y$.

Assume that $y \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$ and so we have $\rho_i(y) \neq \emptyset$ for every $i \in I$. Because of the fact that$\{(\rho_i, W_i) : i \in I\}$ is a soft Γ-semiring over S for every $i \in I$, we have that $\rho_i(y)$ is a sub-Γ-semiring over S for every $i \in I$. It follows that $\psi(y) = \bigcap_{i\in I}\rho_i(y)$ is a sub-Γ-semiring over S for every $y \in (\psi, Y)$. Thus, $\left(\widetilde{\cap}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i)$ is a soft-Γ-semiring over S. ■

Theorem 3. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semirings over S. If $\rho_i(y_i) \subseteq \rho_j(y_j)$ or $\rho_j(y_j) \subseteq \rho_i(y_i)$ for all $i, j \in I, y_i \in W_i$ then the restricted union $\left(\widetilde{\cup}_{\Re}\right)_{i\in I}(\rho_i, W_i)$ is a soft-Γ-semiring over S.

Proof. Using definition 2.5 (ii), we have that $\left(\widetilde{\cup}_{\Re}\right)_{i\in I}(\rho_i, W_i) = (\psi, Y)$, where $Y = \bigcap_{i\in I} W_i$, and $\psi(y) = \bigcup_{i\in I}\rho_i(y)$ for all $y \in Y$. Assume that $y \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$ and so we have $\rho_{i_0}(y) \neq \emptyset$ for some $i_0 \in I(y)$. By assumption, $\bigcup_{i\in I}\rho_i(y)$ is a sub-Γ-semiring of S for every $y \in Supp(\psi, Y)$. Hence, $\left(\widetilde{\cup}_{\Re}\right)_{i\in I}(\rho_i, W_i)$ is a soft-Γ-semiring over S. ■

Theorem 4. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semiring over S. Let W_i and W_j be members of the family $\{W_i : i \in I\}$ such that $W_i \cap W_j = \emptyset$ for all $i \neq j$. Then $\left(\widetilde{\cup}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i)$ is a soft-Γ-semiring over S.

Proof. From definition 2.6 (ii) we have that where $\left(\widetilde{\cup}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i) = (\psi, Y)$ $\psi(y) = \bigcap_{i\in I}\rho_i(y)$ for all $y \in Y$. Note first that (ψ, Y) is non-null owing to the fact that $Supp(\psi, Y) = \bigcup_{i\in I} Supp(\rho_i, W_i)$. Suppose that $y \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$ so we have $\rho_{i_0} \neq \emptyset$ for some $i_0 \in I(y)$. From the hypothesis $\{W_i : i \in I\}$ are pairwise disjoint, we follow that $\varphi(y) = \rho_{i_0}(y)$. On the other hand $\rho_{i_0}(y)$ is a soft Γ-semiring over S, we conclude that (ψ, Y) is a soft Γ-semiring over S for all $y \in (\psi, Y)$. Consequently $\left(\widetilde{\cup}_{\mathscr{E}}\right)_{i\in I}(\rho_i, W_i) = (\psi, Y)$ is a soft Γ-semiring over S. ■

Theorem 5. If (ρ, W) and (σ, Y) be two soft Γ-semirings over Γ-semiring S, then $(\rho, W)\widetilde{\Lambda}(\sigma, Y)$ is a soft Γ-semiring over S if it is non-null.

Proof. Using definition 2.7 (i) , we have that $(\rho, W)\widetilde{\Lambda}_{\mathscr{E}}(\sigma, Y) = (\psi, Z)$, where $Z = W \times \Gamma \times Y$ and $\psi(\omega, \alpha, y) = \rho(\omega) \cap \sigma(y)$ for all $(\omega, \alpha, y) \in Z = W \times \Gamma \times Y$. Then by the hypothesis, (ψ, Z) is a non-null soft set over Γ-semiring S. Since (ψ, Z) is a nonnull, $Supp(\psi, Z) \neq \emptyset$ and so, for $(\omega, \alpha, y) \in Supp(\psi, Z)$, $\psi(\omega, \alpha, y) = \rho(\omega) \cap \sigma(y) \neq \emptyset$. We assume that $t_1, t_2 \in \rho(\omega) \cap \sigma(y)$. In this position

i) If $t_1, t_2 \in \rho(\omega) = \{y : R(\omega, \alpha_1, y), \forall \alpha_1 \in \Gamma\}$ we have that $\omega \alpha_1 t_1 \in W, \omega \alpha_1 t_2 \in W$. This implies $\omega \alpha_1 (t_1 + t_2) \in W$ and
ii) $t_1, t_2 \in \sigma(y) = \{y' : R(y, \alpha_2, y'), \forall \alpha_2 \in \Gamma\}$ we have that $y \alpha_2 t_1 \in Y, y \alpha_2 t_1 \in Y$. This implies $y \alpha_2 (t_1 + t_2) \in Y$.

Hence $\rho(x) \cap \sigma(y)$ is a sub-Γ semiring. By definition of soft Γ semiring, (ρ, W) and (σ, Y) are both soft Γ semirings over S. $\rho(x)$ and $\sigma(y)$ are also sub-Γ semiring of S. Furthermore $\psi(\omega, \alpha, y) = \rho(\omega) \cap \sigma(y)$ is a sub-Γ semiring of S for all $(\omega, \alpha, y) \in (\psi, Z) = (\rho, W) \widetilde{\wedge} (\sigma, Y)$ is a soft Γ semiring over S required. ■

Theorem 6. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semiring over S. Then $\widetilde{\Lambda}_{i \in I}(\rho_i, W_i)$ is a soft Γ-semiring over S if it is non-null.

Proof. By taking into account to the definition 2.7 (ii) we write $\widetilde{\wedge}_{i \in I}(\rho_i, W_i) = (\psi, Y)$, where $Y = \prod_{i \in I} W_i$, and $\psi(y) = \bigcap_{i \in I} \rho_i(y)$ for all $y = (y_i)_{i \in I} \in Y$.

Suppose that (ψ, Y) is non-null. If $y = (y_i)_{i \in I} \in Supp(\psi, Y)$, then $\psi(y) \neq \emptyset$. Since (ρ_i, W_i) is a soft Γ-semiring over S for all $i \in I$ members of nonempty family $((\rho_i, W_i) : i \in I)$ such that $\rho_i(y_i)$ is a sub-Γ-semiring of S. Hence $\psi(y)$ is a sub-Γ-semiring of S for all $y \in Supp(\psi, Y)$, and so $\widetilde{\wedge}_{i \in I}(\rho_i, W_i) = (\psi, Y)$ is soft Γ-semiring over S. ■

Theorem 7. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semiring over S. If $\rho_i(y_i) \subseteq \rho_j(y_j)$ or $\rho_j(y_j) \subseteq \rho_i(y_i)$ for all $i, j \in I, y_i \in W_i$, the $\vee$-union $\widetilde{\vee}_{i \in I}(\rho_i, W_i)$ is a soft Γ-semiring over S.

Proof. Using the definition 2.8 (ii), we have that $\widetilde{\vee}_{i \in I}(\rho_i, W_i) = (\psi, Y)$ where $Y = \prod_{i \in I} A_i$, and $\psi(y) = \bigcup_{i \in I} \rho_i(y)$ for all $y = (y_i)_i \in I \in Y$.

Assume that $y = (y_i)_i \in I \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$ and so we have that $\rho_{i_0}(y) \neq \emptyset$ for some $i_0 \in I$. By assumption, $\bigcup_{i \in I} \rho_i(y)$ is a soft Γ-semiring of S for all $y = (y_i)_i \in I \in Supp(\psi, Y)$. Consequently $\widetilde{\vee}_{i \in I}(\rho_i, W_i) = (\psi, Y)$ is a soft-Γ-semiring over S ■

Theorem 8. Let $\{(\rho_i, W_i) : i \in I\}$ be a nonempty family of soft Γ-semirings over S_i.Then $\widetilde{\prod}_{i \in I}(\rho_i, W_i)$ is a soft Γ-semiring over $\prod_{i \in I} S_i$.

Proof. By definition 2.10 we write $\widetilde{\prod}_{i \in I}(\rho_i, W_i) = (\psi, Y)$, where $Y = \prod_{i \in I} W_i$, and $\psi(y) = \prod_{i \in I} W_i(y)$ for all $y = (y_i)_{i \in I} \in Y$.

Let $y = (y_i)_{i \in I} \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$, and so we have $\rho_i(y_i) \neq \emptyset$ for all $i \in I$. By taking into account,$\{(\rho_i, W_i) : i \in I\}$ is a soft Γ-semiring over S_i for all $i \in I$, it follows that $\prod_{i \in I} \rho_i(y_i)$ is a soft-Γ-semiring of $\prod_{i \in I} S_i$ for all $y = (y_i)_{i \in I} \in Supp(\psi, Y)$.Hence $\widetilde{\prod}_{i \in I}(\rho_i, W_i)$ is a soft Γ-semiring over $\prod_{i \in I} S_i$ ■

Definition 16. Let (ρ, W) be soft Γ-semiring over S.

i) (ρ, W) is called the trivial soft Γ-semiring over S if $\rho(\omega) = \{0\}$ for all $\omega \in W$
ii) (ρ, W) is called the whole soft Γ-semiring over S if $\rho(\omega) = S$ for all $\omega \in W$

Definition 17. Let S and S' be two Γ-semiring and $f : S \to S'$ a mapping of Γ-semiring. If (ρ, W) and (σ, Y) are soft sets over S and S' respectively, then

i) $(f(\rho), W)$ is a soft set over S' where

$$f(\rho): W \to P(S')$$

$$f(\rho)(\omega) = f(\rho(w))$$

for all $\omega \in W$.

ii) $(f^{-1}(\sigma), Y)$ is a soft set over S where

$$f^{-1}(\sigma): Y \to P(S)$$

$$f^{-1}(\sigma)(y) = f^{-1}(\sigma(y))$$

for all $y \in Y$.

Lemma 2. Let $f: S \to S'$ be an onto homomorphism of Γ-semiring. The following statements can be given.

i) (ρ, W) be soft Γ-semiring over S, then $(f(\rho), W)$ is a soft Γ-semiring over S'
ii) (σ, Y) be soft Γ-semiring over S, then $(f^{-1}(\sigma), Y)$ is a soft Γ-semiring over S.

Proof. i) Since (ρ, W) is a soft Γ-semiring over S, it is clear that $(f(\rho), W)$ is a non-null soft set over S'. For every $y \in Supp(f(\rho), W)$ we have $f(\rho)(y) = f(\rho(y)) \neq \emptyset$. Hence $f(\rho(y))$ which is the onto homomorphic image of Γ-semiring $\rho(y)$ is a Γ-semiring of S' for all $y \in Supp(\rho(f), W)$. That is $(f(\rho), W)$ is a soft Γ-semiring of S'.

ii) It is easy to see that $Supp(f^{-1}(\sigma), Y) \subseteq Supp(\sigma, Y)$. By this way let $y \in Supp(f^{-1}(\sigma), Y)$. Then $\sigma(y) \neq \emptyset$. Hence $f^{-1}(\sigma(y))$ which is homomorphic inverse image of Γ-semiring $\sigma(y)$, is a soft Γ-semiring over S for all $y \in Y$.

■

Theorem 9. Let $f: S \to S'$ be a homomorphism of Γ-semiring and (ρ, W) and (σ, Y) be two soft Γ-semiring over S and S', respectively. Then the following statements can be given.

i) If $\rho(\omega) = ker(f)$ for all $\omega \in W$, then $(f(\rho), W)$ is the trivial soft Γ-semiring over S'.
ii) If f is onto and (ρ, W) is whole, then $(f(\rho), W)$ is the whole soft Γ-semiring over S'.
iii) If $\sigma(y) = f(S)$ for all $y \in Y$, then $(f^{-1}(\sigma), Y)$ is the whole soft Γ-semiring over S.
iv) If f is injective and (σ, Y) is trivial, then $(f^{-1}(\sigma), Y)$ is the trivial soft Γ-semiring over S.

Proof. i) By using $\rho(\omega) = ker(f)$ for all $y \in W$. Then $f(\rho)(\omega) = f(\rho(\omega)) = \{0_{S'}\}$ for all $\omega \in W$. Hence $(f(\rho), W)$ is soft Γ-semiring over S' by Lemma 3.16 and Definition 3.14.

ii) Suppose that f is onto and (ρ, W) is whole. Then $\rho(\omega) = S$ for all $\omega \in W$, and so $f(\rho)(\omega) = f(\rho(\omega)) = f(S) = S'$ for all $\omega \in W$. It follows from Lemma 3.16 and Definition 3.14 that $(f(\rho), W)$ is the whole soft Γ-semiring over S'.

iii) If we use hypothesis $\sigma(y) = f(S)$ for all $y \in Y$, we can write $f^{-1}(\sigma)(y) = f^{-1}(\sigma(y)) = f^{-1}(f(S)) = S$ for all $y \in Y$. It is clear that, $(f^{-1}(\sigma), B)$ is the whole soft Γ-semiring over S by Lemma 3.16 and Definition 3.14.

iv) Suppose that f is injective and (σ, Y) is trivial. Then, $\sigma(y) = \{0\}$ for all $y \in Y$, so $f^{-1}(\sigma)(y) = f^{-1}(\sigma(y)) = f^{-1}(\{0\}) = kerf = \{0_S\}$ for all $y \in Y$. It follows from Lemma 3.16 and Definition 3.14 that $(f^{-1}(\sigma), Y)$ is the trivial soft Γ-semiring over S.

■

4. Soft Sub-Γ-Semiring

Definition 18. Let (ρ, W) and (σ, Y) be two soft Γ-semirings over S. Then the soft Γ-semiring is called a soft sub Γ-semiring of (ρ, W), denoted by $(\sigma, Y) \subset_{\Gamma_s} (\rho, W)$, if it satisfies the following conditions

i) $Y \subseteq W$,

ii) $\sigma(y)$ is a sub-Γ-Semiring of $\rho(y)$ for all $y \in Supp(\sigma, Y)$.

From the above definition, it is easily deduced that if (σ, Y) is a soft sub-Γ-Semiring of (ρ, W), then $Supp(\sigma, Y) \subset Supp(\rho, W)$.

Theorem 10. Let (ρ, W) and (σ, Y) be two soft Γ-semirings over S and $(\rho, W) \widetilde{\subseteq} (\sigma, Y)$. Then $(\sigma, Y) \subset_{\Gamma_s} (\rho, W)$,

Proof. Straightforward. ■

Theorem 11. Let (ρ, W) and (σ, Y) be two soft Γ-semirings over S and $(\rho, W) \widetilde{\sqcap} (\sigma, Y)$ is a soft sub-Γ semiring of both (ρ, W) and (σ, Y) if it is non-null.

Proof. Straightforward. ■

Theorem 12. Let (ρ, W) be soft Γ-semiring over S and $\{(\psi_i, W_i) : i \in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ, W). Then the restricted intersection $\left(\widetilde{\cap}_{\Re}\right)_{i \in I} (\psi_i, W_i)$ is a soft sub-Γ-semiring of (ρ, W) if it is non-null.

Proof. Similar to the proof of Theorem 3.6. ■

Corollary 2. Let (ρ, W) be soft Γ-semiring over S and $\{(\psi_i, W) : i \in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ, W). Then $\left(\widetilde{\cap}_{\Re}\right)_{i \in I} (\psi_i, W)$ is a soft sub-Γ-semiring of (ρ, W) if it is non-null.

Proof. Straightforward. ■

Theorem 13. Let (ρ, W) be soft Γ-semiring over S and $\{(\psi_i, W_i) : i \in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ, W). Then the extended intersection $\left(\widetilde{\cap}_E\right)_{i \in I} (\psi_i, W_i)$ is a soft sub-Γ-semiring of (ρ, W).

Proof. Similar to the proof of Theorem 3.7. ■

Theorem 14. Let (ρ, W) be soft Γ-semiring over S and $\{(\psi_i, W_i) : i \in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ, W). If $\psi_i(y_i) \subseteq \psi_j(y_j)$ or $\psi_j(y_j) \subseteq \psi_i(y_i)$ for all $i, j \in I, y_i \in W_i$, then the restricted union $\left(\widetilde{\cup}_{\Re}\right)_{i \in I} (\psi_i, W_i)$ is a soft sub-Γ-semiring of (ρ, W).

Proof. By the aid of the definition 2.6 (ii), we write $\left(\widetilde{\cup}_{\mathscr{E}}\right)_{i \in I} (\psi_i, W_i) = (\psi, Y)$, where $Y = \bigcup_{i \in I} W_i$, and $\psi(y) = \bigcup_{i \in I} \psi_i(y)$ for all $y \in Y$.

Let $y \in Supp(\psi, Y)$. Then $\psi(y) \neq \emptyset$, and so we have $\psi_{i_0}\left(y_{i_0}\right) \neq \emptyset$ for some $i_0 \in I$.From the hypothesis, we know that $\psi_i(y_i) \subseteq \psi_j(y_j)$ or $\psi_j(y_j) \subseteq \psi_i(y_i)$ for all $i, j \in I, y_i \in W_i$, clearly $\bigcup_{i \in I} \psi_i(y)$ is a sub-Γ-semiring of $\rho(y)$ for all $y \in Supp(\psi, Y)$. Thus $\left(\widetilde{\cup}_{\Re}\right)_{i \in I} (\psi_i, W_i) = (\psi, Y)$ is a soft sub-Γ-semiring of (ρ, W). ■

Theorem 15. Let (ρ, W) be soft Γ-semiring over S and $\{(\psi_i, W_i) : i \in I\}$ be nonempty family of soft sub-Γ-semiring of (ρ, W). If $\psi_i(y_i) \subseteq \psi_j(y_j)$ or $\psi_j(y_j) \subseteq \psi_i(y_i)$ for all $i, j \in I, y_i \in W_i$,then $\vee$ union $\widetilde{\vee}_{i \in I} (\psi_i, W_i)$ is a soft sub-Γ-semiring of $\widetilde{\vee}_{i \in I} (\rho, W)$.

Proof. Similar to the proof of Theorem 3.12. ■

Theorem 16. Let (ρ,W) be a soft Γ-semiring over S and $\{(\psi_i,W_i): i\in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ,W). Then the $\wedge$ intersection $\widetilde{\wedge}_{i\in I}(\psi_i,W_i)$ is a soft sub-Γ-semiring of $\widetilde{\wedge}_{i\in I}(\rho,W)$.

Proof. Similar to the proof of Theorem 3.11. ■

Theorem 17. Let (ρ,W) be soft Γ-semiring over S and $\{(\psi_i,W_i): i\in I\}$ be nonempty family of soft sub-Γ-semirings of (ρ,W). Then the cartesian product of the family $\widetilde{\prod}_{i\in I}(\psi_i,W_i)$ is a soft sub-Γ-semiring of $\widetilde{\prod}_{i\in I}(\rho,W)$.

Proof. By Definition 2.10, we can write $\widetilde{\prod}_{i\in I}(\psi_i,W_i)=(\psi,Y)$ where $Y=\prod_{i\in I}W_i$ and $\psi(y)=\prod_{i\in I}\psi_i(y_i)$ for all $y=(y_i)_{i\in I}\in Y$. Let $y=(y_i)_{i\in I}\in Supp(\psi,Y)$. Then $\psi(y)\neq\emptyset$ and so we have $\psi_i(y_i)\neq\emptyset$ for all $i\in I$. In as much as $\{(\psi_i,W_i): i\in I\}$ is a soft sub-Γ-semiring of (ψ,W), we have that $\psi_i(y_i)$ is a sub-Γ-semiring of $\rho(y_i)$. It follows that, we obtain $\prod_{i\in I}\psi_i(y_i)$ for all $y=(y_i)_{i\in I}\in Supp(\psi,Y)$. Hence, the cartesian product of the family $\widetilde{\prod}_{i\in I}(\rho_i,W_i)$ is a soft sub-Γ-semiring of (ρ,W). ■

Theorem 18. Let $f:S\to S'$ be a homomorphism of Γ-semirings and (ρ,W) and (σ,Y) be two soft Γ-semirings over S. If $(\sigma,Y)\subset_{\Gamma_S}(\rho,W)$ then $(f(\sigma),Y)\subset_{\Gamma_S}(f(\rho),W)$.

Proof. Suppose that $y\in Supp(\sigma,Y)$. Then $y\in Supp(\rho,W)$. By definition 4.1, we know that $Y\subseteq W$ and $\sigma(y)$ is a sub Γ-semiring of $\rho(y)$ for all $y\in Supp(\sigma,Y)$. From the expression hypothesis f is a homomorphism, $f(\sigma)(y)=f(\sigma(y))$ is a sub-Γ-semiring of $f(\rho)(y)=f(\rho(y))$ and therefore $(f(\sigma),Y)\subset_{\Gamma_S}(f(\rho),W)$. ■

Theorem 19. Let $f:S\to S'$ be a homomorphism of Γ-semiring and (ρ,W), (σ,Y) be two soft Γ-semirings over S. If $(\sigma,Y)\subset_{\Gamma_S}(\rho,W)$ then $\left(f^{-1}(\sigma),Y\right)\subset_{\Gamma_S}\left(f^{-1}(\rho),W\right)$.

Proof. Let $y\in Supp\left(f^{-1}(\sigma),Y\right)$. $Y\subseteq W$ and $\sigma(y)$ is a sub Γ-semiring of $\rho(y)$ for all $y\in Y$. Since f is a homomorphism, $f^{-1}(\sigma)(y)=f^{-1}(\sigma(y))$ is a sub Γ-semiring of $f^{-1}(\sigma(y))=f(\sigma)(y)$ for all $y\in Supp\left(f^{-1}(\sigma),Y\right)$. Hence $\left(f^{-1}(\sigma),Y\right)\subset_{\Gamma_S}\left(f^{-1}(\rho),W\right)$ ■

5. Homomorphisms of Soft Γ-semiring

Definition 19. Let (ρ,W) and (σ,Y) be two soft Γ-semiring over S and S', respectively. Let $\phi:S\to S'$ and $\theta:W\to Y$ be two functions. Then (ϕ,θ) is called a soft Γ-homomorphism from (ρ,W) to (σ,Y). The latter is written by $(\rho,W)\sim_{\Gamma_S}(\sigma,Y)$ if the following conditions are satisfied:

i) ϕ is an epimorphism of Γ-semiring,
ii) θ is a surjective mapping, and
iii) $\phi(\rho(w))=\sigma(\theta(x))$ for all $w\in W$.

In the above definition, if θ is a Γ-isomorphism from S to S' and θ is a bijective mapping, then (ϕ,θ) is called a Γ-isomorphism so that (ϕ,θ) is a soft Γ-isomorphism from (ρ,W) to (σ,Y), denoted by $(\rho,W)\simeq_{\Gamma_S}(\sigma,Y)$.

By utilizing the definition of soft Γ-homomorphism, we can also obtain some properties of soft Γ-homomorphisms having similar properties as the classical homomorphisms.

An equivalence relation ξ on Γ-semiring S is called a congruence if for all $x,y,z \in S$ and $\gamma \in \Gamma$, we get

$$x\zeta y \Rightarrow (x+z)\,\zeta\,(y+z)$$
$$x\zeta y \Rightarrow (x\gamma z)\,\zeta\,(y\gamma z) \text{ and } (z\gamma x)\,\zeta\,(z\gamma y)$$

The set of all equivalence classes of the elements of S with respect to the relation ξ, that is

$$S_\zeta = \{\zeta(x) \mid x \in S\}.$$

[26]

Proposition 2. Let (ρ,W) be soft Γ-semiring over S and (σ,Y) be a soft set over Γ-semiring over S'. If (ϕ,θ) is a soft Γ-homomorphism from (ρ,W) to (σ,Y), then (σ,Y) is also a soft Γ-semiring over S'.

Proof. Since (ϕ,θ) is a soft Γ-semiring from (ρ,W) to (σ,Y) and (ρ,W) is a soft Γ-semiring over S, then $\phi(S) = S'$ is a Γ-semiring. Now for all $y \in Y$, there exists $w \in W$ such that $\theta(w) = y$. Thus $\sigma(y) = \sigma(\theta(w)) = \phi(\rho(w))$ is a sub-Γ-semiring of S'. Hence (σ,Y) is a soft Γ-semiring over S'. ■

Proposition 3. Let (ρ,W) and (σ,Y) be soft Γ-semirings over S and S', respectively. If (ϕ,θ) is a soft Γ-homomorphism from (ρ,W) to (σ,Y) and (ρ_1,W_1) is a soft sub-Γ-semiring of (ρ,W), then $(\sigma,\theta(W_1))$ is a soft sub-Γ-semiring of (σ,Y).

Proof. Since (ϕ,θ) is a soft Γ-homomorphism from (ρ,W) to (σ,Y) and (ρ_1,W_1) is a soft sub-Γ-semiring of (ρ,W), we get $\theta(W_1) \subset Y$ and (σ,w) is a soft sub-Γ-semiring of S' for all $w \in \theta(W_1)$. It means that $(\sigma,\theta(W_1))$ is a soft sub-Γ-semiring of (σ,Y) ■

Now, we construct the following three Γ-isomorphism theorems for soft Γ-semirings.

Theorem 20 (First Γ-Isomorphism Theorem)**.** Let (ρ,W) and (σ,Y) be soft Γ-semirings over S and S', respectively. If (ϕ,θ) is a soft Γ-homomorphism from (ρ,W) to (σ,Y) and $\rho(w) \supset Ker\phi$ for all $w \in W$. Then the following conditions hold:

1) $(I,W) \simeq_{\Gamma_S} (J,W)$, where $(I,w) \simeq_{\Gamma_S} (\sigma,w)/Ker\phi,\ (J,w) = \phi(\rho(w)),\ w \in W$,
2) If θ is a bijective mapping, then $(I,W) \simeq_{\Gamma_S} (\sigma,Y)$

Proof. (1) It is trivial that $Ker\phi$ is an ideal of S so that $S/Ker\phi$ is a Γ-semiring. Also $Ker\phi$ is an ideal of $\rho(w)$ and $\rho(w)/Ker\phi$ is a Γ-semiring for all $w \in W$. Furthermore, $\rho(w)/Ker\phi$ is always an ideal of $S/Ker\phi$ which implies that (I,W) is a soft Γ-semiring over $S/Ker\phi$. We can easily see that $(J,w) = \phi(\rho(w)) = \sigma(\theta(w))$ is an ideal S'. Therefore (J,W) is a soft Γ-semiring over S'.

Let $\overline{\phi} : S/Ker\phi \longrightarrow S'$ be defined by $\overline{\phi}(S+Ker\phi) = \phi(s)$ for all $s \in S$. It is clear that $\overline{\phi}$ is a bijective mapping from $S/Ker\phi$ to S'. For all $s_1,s_2 \in S$ and $\alpha \in \Gamma$

$$\begin{aligned}
\overline{\phi}((s_1+Ker\phi)+(s_2+Ker\phi)) &= \overline{\phi}((s_1+s_2)+Ker\phi) = \phi(s_1+s_2)\\
&= \phi(s_1)+\phi(s_2)\\
&= \overline{\phi}(s_1+Ker\phi)+\overline{\phi}(s_2+Ker\phi)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\phi}((s_1+Ker\phi)\,\alpha\,(s_2+Ker\phi)) &= \overline{\phi}((s_1+s_2)+Ker\phi) = \phi(s_1\,\alpha\,s_2)\\
&= \phi(s_1)\,\alpha\,\phi(s_2)\\
&= \overline{\phi}(s_1+Ker\phi)\,\alpha\,\overline{\phi}(s_2+Ker\phi)
\end{aligned}$$

Therefore $\overline{\phi}$ is a Γ-isomorphism.

Define $\overline{\theta}: W \longrightarrow W$ by $\overline{\theta}(w) = w$. Then $\overline{\theta}$ is a bijective mapping from W to W. Then, we get $\overline{\theta}(I(w)) = \overline{\theta}(\rho(w)/Ker\,\phi) = \phi(\rho(w)) = J(w) = J(\overline{\theta}(w))$. Consequently (ϕ,θ) is a soft Γ-homomorphism so that $(I,W) \simeq (J,W)$.

(2) Define $\overline{\theta}: Ker\,\phi \longrightarrow S'$ by $\overline{\theta}(s+Ker\,\phi) = \theta(s)$. Then $\overline{\theta}$ is a Γ-isomorphism from $S/Ker\,\phi$ to S'. Hence, we get $(I,W) \simeq (J,W)$ because θ is a bijective mapping and $\overline{\theta}(I(w)) = \overline{\theta}(\rho(w)/Ker\,\phi) = \phi(\rho(w)) = \sigma(\theta(w))$. ■

Theorem 21. Let (ρ,W) be soft Γ-semiring over S. If (σ,Y) and (I,Z) are soft sub-Γ-semiring of (ρ,W) then $(P,Y) \sim_{\Gamma_S} (Q,Y)$ and $(S,Z) \sim_{\Gamma_S} (T,Z)$ where

$$P(y) = \sigma(y)/(A\cap B), \quad Q(y) = (\sigma(y)+B)/B,$$
$$S(z) = I(z)/(A\cap B), \quad T(z) = (I(z)+A)/A,$$

$A = \bigcap_{y\in Y} \sigma(y)$ and $B = \bigcap_{z\in Z} I(z)$.

Proof. Firstly we write $C =< \bigcap_{y\in Y} \sigma(y) >$ and $D =< \bigcap_{z\in Z} I(z) >$. Then $A = \bigcap_{y\in Y} \sigma(y)$ is a Γ-ideal of S. It is clear that A is also an Γ-ideal of C so that $A\cap B$ is a Γ-ideal of C and hence (P,y) is a soft Γ-semiring over $C/A\cap B$. Also, it is clear that (Q,Y) is a soft Γ-semiring over $(C+B)/B$.

Now, we define $\phi : C/A\cap B \longrightarrow (C+B)/B$ by $\phi(C+(A\cap B)) = C+B$ and define $\theta : Y \longrightarrow Y$ by $\phi(y) = y$. Then ϕ is a Γ-homomorphism from $C/A\cap B$ to $(C+B)/B$, where θ is a bijective mapping and $\phi(P(y)) = \phi(\sigma(y)/(A\cap B)) = (\sigma(y)+B)/B = Q(y) = Q(\theta(y))$. It follows that $(P,Y) \sim_{\Gamma_S} (Q,Y)$. Similarly, $(S,Z) \sim_{\Gamma_S} (T,Z)$ can easily be proved. ■

Theorem 22 (Second Γ-Isomorphism Theorem)**.** Let (ρ,W) be soft Γ-semirings over S. If (σ,Y) and (I,Z) are soft sub-Γ-semiring of (ρ,W) such that $(\sigma,Y) = A$ for all $y \in Y$, then $(P,Y) \simeq_{\Gamma_S} (Q,Y)$, where

$$P(y) = \sigma(y)/(A\cap B),\ Q(y) = (\sigma(y)+B)/B,\ B = \bigcap_{y\in Z} I(y).$$

Proof. The proof is similar to proof of above theorem, so omit the proof. It is trivial that $(P,Y) \simeq_{\Gamma_S} (Q,Y)$. ■

Theorem 23 (Third Γ-Isomorphism Theorem)**.** Let (ρ,W) be soft Γ-semirings over S. If (σ,Y) and (I,Z) are soft sub-Γ-semiring of (ρ,W) with $Y\cap Z \neq \emptyset$ and $I(x) \subset \sigma(x)$ for all $x \in Y\cap Z$, then $(P,Y\cap Z) \simeq_{\Gamma_S} (Q,Y\cap Z)$, where

$$P(x) = (\rho(x)/B)/(A/B),\ Q(x) = \rho(x)/A$$

with

$$A = \bigcap_{x\in Y\cap Z} \sigma(x) \text{ and } B = \bigcap_{x\in Y\cap Z} I(x).$$

Proof. It can be easily showed that $A = \bigcap_{x\in Y\cap Z} \sigma(x)$ and $B = \bigcap_{x\in Y\cap Z} I(x)$ are sub-$\Gamma-$semiring of S, and B is a Γ-ideal of A. Now, it is clear that $(P, Y \cap Z)$ is a soft Γ-semiring over $(S/B)/(A/B)$ and so $(Q, Y \cap Z)$ is a soft Γ-semiring over S/A.

Define the mapping

$$\begin{aligned} \phi : (S/B)/(A/B) &\longrightarrow S/A \\ S+B+(A/B) &\longrightarrow S+A \end{aligned}$$

by $\phi((r+B)+(A/B)) = S+A$ and define

$$\begin{aligned} \theta : Y \cap Z &\longrightarrow Y \cap Z \\ x &\longrightarrow \theta(x) = x \end{aligned}$$

by $\theta(x) = x$. Then ϕ is a Γ-isomorphism from $(S/B)/(A/B)$ to S/A. Obviously, θ is a bijective mapping and

$$\phi(P(x)) = \phi((\rho(x)/B)/(A/B)) = \rho(x)/A = Q(\theta(x)).$$

Finally, $(P, Y \cap Z) \simeq_{\Gamma_S} (Q, Y \cap Z)$, and hence third Γ-isomorphism theorem is proved. ■

6. Conclusion

In this study, we introduced soft Γ-semiring and gave some properties of soft Γ-semiring by giving examples. Then soft sub-Γ-semiring are defined and their properties are proposed. Furthermore, focus on establishing First, Second and Third Isomorphism Theorems of soft Γ-semiring, respectively. To extend this study, one could study other algebraic structures and do some further study on the properties them.

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