Modelling & Application & Theory
ISSN: 2548–0596
Vol. 3, No. 1, 2018, pp. 35-50

How to Determine Properties of the Special Curves Using Real Octonions in Euclidean 8-Space

Ozcan Bektas[1,*], Salim Yuce[2]

[1]*Department of Mathematics, Faculty of Arts and Sciences, Recep Tayyip Erdogan University, Rize, TURKEY*
[2]*Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, Istanbul, TURKEY*

Abstract. In this study, we approach to the issue of how to determine properties of special octonionic curves (octonionic involute-evolute curves, octonionic Bertrand curves, octonionic Smarandache curves, and octonionic Mannheim curves) by means of real octonions in Euclidean 8-space. Firstly, we give some information about octonion algebras, and octonionic curves in Euclidean 8-space. After that, the chapter of the special octonionic curves are divided into four part. Finally, we obtain some characterizations of the special octonionic curves.

Keywords. Serret-Frenet formula, Spatial real octonionic curve, octonionic involute-evolute curve, octonionic Bertrand curve, octonionic Mannheim curve, octonionic Smarandache curve.

2000 MSC. Primary 53A04; 14H50; 11R52.

1. Introduction

Nowadays, the octonion algebras are studied in differential geometry, and clifford algebras.The scientists use especially the octonion algebra in matrix, curve, and surface theories. In the light of these information, we ask a question to ourselves: Can we determine the special octonionic curves, and some characterizations of these curves by using real octonion algebra, and real octonionic curves in Euclidean 8-space? To answer the question, we have to speak of the special curves by using quaternion algebras, and quaternionic curves.

Involute-evolute curves were studied by Ozyilmaz, Yilmaz [27], Turgut and Esin, [31]. Bertrand [7] demonstrated that a necessary and sufficient condition for the existence of such a second curve is required; in fact, a linear relationship computed with constant coefficients should exist between the first and second curvatures of the given original curve. The authors studied the Bertrand curves in papers [10, 11, 15]. Liu and Wang [23] obtained the necessary and sufficient conditions for the Mannheim partner curves in Euclidean space E^3, and Minkowski space E_1^3. Ali [3] defined the special Smarandache curves in the Euclidean space, and determined Serret-Frenet apparatus of these curves.

Quaternions were discovered by Hamilton [21]. Bharathi and Nagaraj [8] computed the Serret-Frenet formulas for quaternionic curves in Euclidean 3-space, and Euclidean 4-space. Then, the authors used the quaternionic curve in differential geometry.

Soyfidan [30] determined the quaternionic involute-evolute curve, and gave some properties about these curves. Kecilioglu and Ilarslan [22] defined $(1,3)$ type Bertrand curves for quatenionic curve in Euclidean 4-space. They proved that if bitorsion of a quatenionic curve α does not vanish, then there is no quaternionic curve in Euclidean 4-space which is a Bertrand curve. Gok and Kahraman [19] constructed quaternionic Bertrand curves in Euclidean 4-space. Okuyucu [26] considered a quaternionic Mannheim curve, and gave

Received 28.11.2018.
*Corresponding author (E-mail): ozcan.bektas@erdogan.edu.tr

some characterizations of them in Euclidean 3-space, and Euclidean 4-space. Cetin and Kocayigit [13] introduced the quaternionic Smarandache curves in 3-dimensional Euclidean space.

The octonions can be thought of as octal of real numbers. Real, and complex numbers, quaternions, and octonions are the four normed division algebras [12]. The octonions have got a lot of properties in [1, 2, 12, 29]. Octonions are non-commutative, and non-associative algebra in mathematics. The octonion is the one of the highest normed division algebra. Moreover, the octonion analysis is largely determined by Baez [4].

Now, we can answer the questions in the first paragraph. Special curves (involute-evolute, Bertrand, Mannheim, and Smarandache) in Euclidean 8-space can be determined by using real octonion. We will introduce octonionic involute-evolute, octonionic Bertrand, octonionic Mannheim, and octonionic Smarandache curves.

2. Preliminaries

In this part, we denote the spatial octions briefly by O_S, and the real octonions with O. Let us firstly refer fundamental notion on the octonions. The octonion A is written as $A = A_0e_0 + \sum_{i=1}^{7} A_ie_i$, where terms A_i are the real numbers coefficients of the real octonions, and e_i $(i = 1, 2, \ldots, 7)$ are called the unit octonions basis elements, and $e_0 = +1$ is called the scalar element. The rules of the unit octonion basis elements are defined in tabular form, and are serviced as such in Table 1 [17].

Table 1.The Multiplication Table of the Unit Octonion Basis Elements

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The octonion addition, the scalar multiplication, and the octonion multiplication are the operations of the set of octonions. The sum of two elements of O is defined by their sum as elements of $\mathbb{R}^8$. That is, the sum of two octonions are written in the form

$$\begin{aligned} A \pm B &= \sum_{i=0}^{7} (A_i \pm B_i)\, e_i \\ &= (A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4 + A_5e_5 + A_6e_6 + A_7e_7) \\ &\quad \pm (B_0e_0 + B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4 + B_5e_5 + B_6e_6 + B_7e_7)\,. \end{aligned}$$

The set of the octonions O is stated by $e_0 \in \mathbb{R}$, and the seven unit octonion basis elements $e_1, e_2, e_3, e_4, e_5, e_6, e_7$; all these elements' square are -1 [17], hence we can write the set of the octonions O as follows $O = \mathbb{R} \oplus \mathbb{R}^7$ [25]. Consequently, we can say that the octonions are isomorphic to $\mathbb{R}^8$ [32]. The points in $\mathbb{R}^8$ are represented by the octonions [4]. $\overline{A}$ is called conjugate of the octonion A, and is defined by

$$\begin{aligned}\overline{A} &= A_0e_0 - A_1e_1 - A_2e_2 - A_3e_3 - A_4e_4 - A_5e_5 - A_6e_6 - A_7e_7 \\ &= A_0e_0 - \sum_{i=1}^{7} A_ie_i,\end{aligned}$$

where $\overline{e_0} = e_0$, and $\overline{e_j} = -e_j$ $(j = 1, \ldots, 7)$ [16]. The octonion A has real part, and vectorial part as well. Therefore, the octonion A is separated related to its real (S_A) in $\mathbb{R}$ and vectorial (V_A) in $\mathbb{R}^7$ parts as follows:

$$S_A = \frac{1}{2}\left(A + \overline{A}\right) = A_0,\ V_A = \frac{1}{2}\left(A - \overline{A}\right) = \sum_{i=1}^{7} A_ie_i.$$

Thus, an octonion is given by $A = S_A + V_A$. The multiplication of two octonions is introduced by

$$A \times B = S_AS_B - \langle V_A, V_B\rangle + S_AV_B + S_BV_A + V_A \wedge V_B$$

$\forall A, B \in O$. Here, the inner product, and the cross product in $\mathbb{R}^7$ are evaluated, respectively [24]. The symmetric non-degenerate reel valued bilinear form g is given by

$$g : O \times O \to \mathbb{R},\ g(A,B) = \frac{1}{2}\left(A \times \overline{B} + B \times \overline{A}\right)$$

$\forall A, B \in O$. g is determined with the help of the octonionic multiplication. g is called the octonionic inner product. Thus, we get $g(A,B) = \sum\limits_{i=1}^{7} A_iBi$ [9,14]. If $A + \overline{A} = 0$, then the octonion A is called spatial octonion. The spatial octonion set is represented by $O_S = \{\sum\limits_{i=1}^{7} A_ie_i;\ A_i \in \mathbb{R}\}$. Here, $e_i^2 = -1$, $e_ie_j = -\delta_{ij}e_0 + \varepsilon_{ijk}e_k$, $(i, j, k = 1, 2, ..., 7)$, $(i \neq j \neq k,\ i \neq 0,\ j \neq 0,\ k \neq 0)$. The spatial octonions are isomorphic to $\mathbb{R}^7$.

The vector product of two vectors is only defined in 3- dimensional Euclidean space, $\mathbb{R}^3$, and 7- dimensional Euclidean space, $\mathbb{R}^7$. We express the vector product in $\mathbb{R}^7$. Let A, B be the spatial octonions. The vector product in $\mathbb{R}^7$ is defined by $A \wedge B = AB + \langle A, B\rangle$ [16,20]. Moreover, this is given by [16, 28] for $A = \sum\limits_{i=1}^{7} A_ie_i = (A_i)$, $1 \leq i \leq 7$, and $B = \sum\limits_{i=1}^{7} B_ie_i = (B_i)$, $1 \leq i \leq 7$. The norm of the octonion A is denoted by

$$\|A\| = \sqrt{A \times \overline{A}} = \sqrt{\sum_{i=0}^{7} A_i^2}.$$

If $\|A_0\| = 1$, then A_0 is called unit octonion.

Inverse properties:

Let $A,\ B$ be two octonions in O. In this condition, the properties $B \times \left(A^{-1} \times A\right) = B$ and $A^{-1} \times (A \times B) = B$ are satified [33]. Let A and B be unit octonions. Since the definition of norm of octonion, we have $B \times \left(\overline{A} \times A\right) = B$ and $\overline{A} \times (A \times B) = B$.

Moufang implies alternative: A Moufang loop is an alternative loop, i.e., it satisfies

$A \times (A \times B) = (A \times A) \times B$ left alternative identity

$(A \times B) \times B = A \times (B \times B)$ right alternative identity

$(A \times B) \times A = (A \times B) \times A$ flexible identity, for all $A,\ B \in O$ [33].

The Serret-Frenet frame and curvatures in $\mathbb{R}^8$: Let $\Gamma : I \subset \mathbb{R} \to \mathbb{R}^8$ be an unit speed space curve in $\mathbb{R}^8$, and $\{\mathbf{U}_j\}$, $1 \leq j \leq 8$ be the Serret-Frenet 8- frame related to Γ. The Serret-Frenet formulas for the curve $\Gamma : I \subset \mathbb{R} \to \mathbb{R}^8$ are given as follows:

$$\begin{array}{l} \mathbf{U}_1'(s) = k_1(s)\,\mathbf{U_2}(s) \\ \mathbf{U}_{\mathbf{m}}'(s) = -k_{m-1}(s)\,\mathbf{U}_{m-1}(s) + k_{m+1}(s)\,\mathbf{U}_{m+\mathbf{1}}(s),\ 2 \le m \le 7 \\ \mathbf{U}_{\mathbf{8}}'(s) = -k_7(s)\,\mathbf{U_7}(s). \end{array}$$

On the other hand, $\mathbf{U}_j(s) = \frac{\mathbf{E}_j(s)}{\|\mathbf{E}_j(s)\|}$, $k_j(s) = \left\langle \mathbf{U}_j'(s), \mathbf{U}_{j+1}(s) \right\rangle = \frac{\|\mathbf{E}_j+1(s)\|}{\|\mathbf{E}_j(s)\|}$ for $1 \le j \le 8$, where $\mathbf{E}_1(s) = \Gamma'(s)$,
and $\mathbf{E}_j(s) = \Gamma^{(j)}(s) - \sum\limits_{i<j} \left\langle \Gamma^{(j)}(s), \mathbf{U}_i(s) \right\rangle \mathbf{U}_j(s)$ [18]. These concepts were transferred to octonion algebra by Bektas and Yuce as follows [5,6]:

Definition 1. Let $\mathbb{R}^7$ characterize a Euclidean space of seven dimensional with octonionic metric g, and $O_S = \left\{ \gamma_O \in O \;\middle|\; \gamma_O + \bar{\gamma}_O = 0 \right\}$ show the spatial octonion set. $\mathbb{R}^7$ is assimilated into the space of the spatial octonion. The curve $\gamma_O : I \subset \mathbb{R} \to O_S$, $\gamma_O(s) = \sum\limits_{i=1}^{7} \gamma_i(s)\, e_i$ is called spatial real octonionic curve (SROC) [5,6].

Theorem 1. Let γ_O be an unit speed spatial octonionic curves (USSROC), and $\{\mathbf{V}_j\}$, $0 \le j \le 6$ be the Serret-Frenet frame of USSROC in $\mathbb{R}^7$. Then Serret-Frenet equations are given by

$$\begin{array}{l} \mathbf{V}_0'(s) = k_1(s)\,\mathbf{V_1}(s) \\ \mathbf{V}_{\mathbf{m}}'(s) = -k_m(s)\,\mathbf{V}_{m-1}(s) + k_{m+1}(s)\,\mathbf{V}_{m+\mathbf{1}}(s) \\ \mathbf{V}_{\mathbf{6}}'(s) = -k_6(s)\,\mathbf{V_5}(s), \end{array} \quad (1)$$

where $\{\mathbf{V}_j\}$ are defined by the following table

Table 2.The Multiplication Table of the Serret-Frenet frame of USSROC in $\mathbb{R}^7$

$\times$	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_0	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	V_1	$-V_0$	V_3	$-V_2$	V_5	$-V_4$	$-V_7$	V_6
V_2	V_2	$-V_3$	$-V_0$	V_1	V_6	V_7	$-V_4$	$-V_5$
V_3	V_3	V_2	$-V_1$	$-V_0$	V_7	$-V_6$	V_5	$-V_4$
V_4	V_4	$-V_5$	$-V_6$	$-V_7$	$-V_0$	V_1	V_2	V_3
V_5	V_5	V_4	$-V_7$	V_6	$-V_1$	$-V_0$	$-V_3$	V_2
V_6	V_6	V_7	V_4	$-V_5$	$-V_2$	V_3	$-V_0$	$-V_1$
V_7	V_7	$-V_6$	V_5	V_4	$-V_3$	$-V_2$	V_1	$-V_0$

and

$k_i, 1 \le i \le 6$, $1 \le m \le 5$ curvature functions. Eq. (1) is called Serret-Frenet formulas of the USSROC [5,6].

Definition 2. Let $\mathbb{R}^8$ characterize a Euclidean space of eight dimensional with octonionic metric g, and $\mathbb{R}^8$ is assimilated into the space of the octonion. The curve $\beta_O : I \subset \mathbb{R} \to O$, $\beta_O(s) = \sum\limits_{i=0}^{7} \gamma_i(s)\, e_i$ is called real octonionic curve (ROC). Note that the vector part of β_O is same to SROC, γ_O in O_S [5,6].

Theorem 2. Let β_O be an USROC, and $\{\mathbf{W}_j\}$, $0 \leq j \leq 7$ be the Serret-Frenet frame of USROC in $\mathbb{R}^8$. Then Serret-Frenet equations are given by [5,6]

$$\begin{aligned}
\mathbf{W}_0'(s) &= K(s)\mathbf{W_1}(s) \\
\mathbf{W_1}'(s) &= -K(s)\mathbf{W}_0(s) + k_1(s)\mathbf{W_2}(s) \\
\mathbf{W_2}'(s) &= -k_1(s)\mathbf{W_1}(s) + (k_2 - K)(s)\mathbf{W_3}(s) \\
\mathbf{W_3}'(s) &= -(k_2 - K)(s)\mathbf{W_2}(s) + k_3(s)\mathbf{W_4}(s) \\
\mathbf{W_4}'(s) &= -k_3(s)\mathbf{W_3}(s) + (k_4 - K)(s)\mathbf{W_5}(s) \\
\mathbf{W_5}'(s) &= -(k_4 - K)(s)\mathbf{W_4}(s) + k_5(s)\mathbf{W_6}(s) \\
\mathbf{W_6}'(s) &= -k_5(s)\mathbf{W_5}(s) + (k_6 + K)(s)\mathbf{W_7}(s) \\
\mathbf{W_7}'(s) &= -(k_6 + K)(s)\mathbf{W_6}(s),
\end{aligned} \tag{2}$$

where $\mathbf{W_1} = \mathbf{V}_0 \times \mathbf{W}_0$, $\mathbf{W_2} = \mathbf{V_1} \times \mathbf{W}_0$, $\mathbf{W_3} = \mathbf{V_2} \times \mathbf{W}_0$, $\mathbf{W_4} = \mathbf{V_3} \times \mathbf{W}_0$,
$\mathbf{W_5} = \mathbf{V_4} \times \mathbf{W}_0$, $\mathbf{W_6} = \mathbf{V_5} \times \mathbf{W}_0$, $\mathbf{W_7} = \mathbf{V_6} \times \mathbf{W}_0$, $K = \left\| \mathbf{W}_0'(s) \right\|$.

3. Special Octonionic Curves

This section is the original part of our paper. Our purpose is to give special curves (involute-evolute, Bertrand, Mannheim, and Smarandache) in Euclidean 8-space can be determined by using real octonion. In addition, we introduce octonionic involute-evolute, octonionic Bertrand, octonionic Mannheim, and octonionic Smarandache curves in this section.

Definition 3. Let α, $\psi : I \subset \mathbb{R} \to O_S$ be any regular real spatial octonionic curves with parameter s^*, and s, respectively. Moreover,
$\{\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5, \mathbf{V}_6\}$, and $\{\mathbf{V}_0^*, \mathbf{V}_1^*, \mathbf{V}_2^*, \mathbf{V}_3^*, \mathbf{V}_4^*, \mathbf{V}_5^*, \mathbf{V}_6^*\}$ denote the Serret-Frenet frame of the curves α, and ψ, respectively.

I. $\{\alpha, \psi\}$ is called spatial octonionic involute-evolute curve couple in O_S if $g(\mathbf{V}_0(s^*), \mathbf{V}_0^*(s)) = 0$.

II. ψ is called spatial octonionic Bertrand curve, and α is called spatial octonionic Bertrand couple of the curve ψ if first principal normal vector fields $\mathbf{V}_1^*(s)$, and $\mathbf{V}_1(s^*)$ of the curves ψ, and α are linearly dependent.

III. $\{\psi,\alpha\}$ is called spatial octonionic Mannheim curve couple in O_S if first principal normal vector $\mathbf{V}_1^*(s)$ of ψ, and second principal normal vector of $\mathbf{V}_2(s^*)$ of α are linearly dependent.

IV. Let $\psi : I \subset \mathbb{R} \to O_S$ be unit speed spatial octonionic curve in Euclidean 7-space. Spatial octonionic $\mathbf{V}_0^*\mathbf{V}_1^* \cdots \mathbf{V}_6^*$-Smarandache curve is defined by

$$\Phi(s^*) = \frac{1}{\sqrt{7}} \left(\sum_{i=1}^{6} \mathbf{V}_i^*(s) \right),$$

where s^* is arc length parameter of spatial octonionic
$\mathbf{V}_0^*\mathbf{V}_1^* \cdots \mathbf{V}_6^*$-Smarandache curve Φ. According to this definition using the Serret-Frenet apparatus, we define 5-Type of spatial octonionic $\mathbf{V}_0^* \cdots \mathbf{V}_j^*$-Smarandache curve, $\mathbf{1} \leq \mathbf{j} \leq 5$ expect of the above spatial octonionic $\mathbf{V}_0^*\mathbf{V}_1^* \cdots \mathbf{V}_6^*$-Smarandache curve.

Definition 4. Let δ,$\eta : I \subset \mathbb{R} \to O$ be any regular real octonionic curves with parameter s^*, and s, respectively. Moreover,
$\{\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4, \mathbf{W}_5, \mathbf{W}_6, \mathbf{W}_7\}$, and
$\{\mathbf{W}_0^*, \mathbf{W}_1^*, \mathbf{W}_2^*, \mathbf{W}_3^*, \mathbf{W}_4^*, \mathbf{W}_5^*, \mathbf{W}_6^*, \mathbf{W}_7^*\}$ denote the Serret-Frenet frame of the curves δ, and η, respectively.

I. $\{\delta,\eta\}$ is called octonionic involute-evolute curve couple in O if
$g(\mathbf{W}_0(s^*),\mathbf{W}_0^*(s)) = 0.$

II. η is called octonionic Bertrand curve, and δ is called octonionic Bertrand couple of the curve η if first principal normal vector fields $\mathbf{W}_1^*(s)$, and $\mathbf{W}_1(s^*)$ of the curves η, and δ are linearly dependent.

III. $\{\eta,\delta\}$ is called octonionic Mannheim curve couple in O_S if first principal normal vector $\mathbf{W}_1^*(s)$ of η, and second principal normal vector $\mathbf{W}_2(s^*)$ of δ are linearly dependent.

IV. Let $\eta : I \subset \mathbb{R} \to O$ be unit speed octonionic curve in Euclidean 8-space. Then octonionic $\mathbf{W}_0^*\mathbf{W}_1^*\cdots\mathbf{W}_7^*$-Smarandache curves is defined by

$$\Lambda(s^*) = \frac{1}{\sqrt{8}}\left(\sum_{i=1}^{7}\mathbf{W}_i^*(s)\right),$$

where s^* is arc length parameter of octonionic $\mathbf{W}_0^*\mathbf{W}_1^*\cdots\mathbf{W}_7^*$-
Smarandache curve Λ. According to this definition using the Serret-Frenet apparatus, we define 6-Type of octonionic $\mathbf{W}_0^*\cdots\mathbf{W}_j^*$-Smarandache curve, $\mathbf{1} \leq \mathbf{j} \leq 6$ expect of the above octonionic $\mathbf{W}_0^*\mathbf{W}_1^*\cdots\mathbf{W}_7^*$-Smarandache curve.

Theorem 3. Let α, $\psi : I \subset \mathbb{R} \to O_S$ be spatial octonionic curves with arc-length parameter s^*, and s, respectively.

Case I. If $\{\alpha,\psi\}$ is spatial octonionic involute-evolute curve couple in O_S, then we get $d(\psi(s),\alpha(s^*)) = |r-s|$, where r is real number.

Case II. If $\{\psi,\alpha\}$ is spatial octonionic Bertrand curve couple in O_S, then we get $d(\psi(s),\alpha(s^*)) = |r|$, where r is real number.

Case III. If $\{\psi,\alpha\}$ is called spatial octonionic Mannheim curve couple in O_S, then we get $d(\psi(s),\alpha(s^*)) = |r|$, where r is real number.

Proof. I. Let $\{\alpha,\psi\}$ be spatial octonionic involute-evolute curve couple in O_S. By the definition of spatial octonionic involute-evolute curves, we have

$$\alpha(s^*) = \psi(s) + \lambda(s)\mathbf{V}_0^*(s). \tag{3}$$

By differentiating the Eq. (3) with respect to s, and by using the Serret-Frenet equations given by Eq. (1), we get

$$\frac{d\alpha(s^*)}{ds^*}\frac{ds^*}{ds} = \left(1+\lambda'(s)\right)\mathbf{V}_0^*(s) + \lambda(s)k_1^*(s)\mathbf{V}_1^*(s).$$

If we write $\frac{d\alpha(s^*)}{ds^*} = \mathbf{V}_0(s^*)$, then we obtain

$$\mathbf{V}_0(s^*)\frac{ds^*}{ds} = \left(1+\lambda'(s)\right)\mathbf{V}_0^*(s) + \lambda(s)k_1^*(s)\mathbf{V}_1^*(s).$$

Let take the octonionic inner product of both sides of the last equation with $\mathbf{V}_0(s^*)$, then we can write

$$g\left(\mathbf{V}_0(s^*)\frac{ds^*}{ds},\mathbf{V}_0^*(s)\right) = g\left(\left[\left(1+\lambda'(s)\right)\right]\mathbf{V}_0^*(s) + \lambda(s)k_1^*(s)\mathbf{V}_1^*(s),\mathbf{V}_0^*(s)\right),$$

and thus

$$\begin{aligned} g\left(\mathbf{V}_0(s^*)\frac{ds^*}{ds},\mathbf{V}_0^*(s)\right) &= \frac{1}{2}\{[\left(1+\lambda'(s)\right)\mathbf{V}_0^*(s) \\ &+\lambda(s)k_1^*(s)\mathbf{V}_1^*(s)\times\overline{\mathbf{V}_0^*(s)}] \\ &+[\mathbf{V}_0^*(s)\times\left(1+\lambda'(s)\right)\overline{\mathbf{V}_0^*(s)} \\ &+\lambda(s)k_1^*(s)\overline{\mathbf{V}_1^*(s)}]\}. \end{aligned}$$

If we take into account conjugate of octonion in the last equation, then

$$\begin{aligned} g\left(\mathbf{V}_0(s^*)\frac{ds^*}{ds},\mathbf{V}_0^*(s)\right) &= \frac{1}{2}\{[\left(1+\lambda'(s)\right)\mathbf{V}_0^*(s) \\ &+\lambda(s)k_1^*(s)\mathbf{V}_1^*(s)]\times(-\mathbf{V}_0^*(s)) \\ &+\mathbf{V}_0^*(s)\times[\left(1+\lambda'(s)\right)(-\mathbf{V}_0^*(s)) \\ &+\lambda(s)k_1^*(s)(-\mathbf{V}_1^*(s))]\} \\ &= \frac{1}{2}\{-\left(1+\lambda'(s)\right)(\mathbf{V}_0^*(s)\times\mathbf{V}_0^*(s)) \\ &-\lambda(s)k_1^*(s)(\mathbf{V}_1^*(s)\times\mathbf{V}_0^*(s)) \\ &-\left(1+\lambda'(s)\right)(\mathbf{V}_0^*(s)\times\mathbf{V}_0^*(s)) \\ &-\lambda(s)k_1^*(s)(\mathbf{V}_0^*(s)\times\mathbf{V}_1^*(s))\} \\ &= \frac{1}{2}\{-2\left(1+\lambda'(s)\right)(\mathbf{V}_0^*(s)\times\mathbf{V}_0^*(s))\} \\ &= -\left(1+\lambda'(s)\right)[-g(\mathbf{V}_0^*(s),\mathbf{V}_0^*(s)) \\ &+\mathbf{V}_0^*(s)\wedge\mathbf{V}_0^*(s)] \\ &= 1+\lambda'(s). \end{aligned}$$

Since spatial octonionic curve $\alpha(s^*)$ is spatial octonionic involute curve of spatial octonionic curve $\psi(s)$, that is to say from the definition of spatial octonionic involute-evolute curve couple, then we have

$$g\left(\mathbf{V}_0(s^*)\frac{ds^*}{ds},\mathbf{V}_0^*(s)\right)=\frac{ds^*}{ds}g\left(\mathbf{V}_0(s^*),\mathbf{V}_0^*(s)\right)=0.$$

So, we get

$$1+\lambda'(s)=0. \tag{4}$$

By the last statements, we obtain

$$\lambda(s)=r-s.$$

By the Eqs. (3) and (4), we find

$$d\left(\psi(s),\alpha(s^*)\right)=\left\|\alpha(s^*)-\psi(s)\right\|=\left\|\lambda(s)\mathbf{V}_0^*(s)\right\|.$$

On the other hand, we have

$$\begin{aligned}\|\lambda(s)\mathbf{V}_0^*(s)\|^2 &= \lambda(s)\mathbf{V}_0^*(s)\times\overline{\lambda(s)\mathbf{V}_0^*(s)}\\ &= \lambda^2(s)\left(\mathbf{V}_0^*(s)\times\overline{\mathbf{V}_0^*(s)}\right)\\ &= -\lambda^2(s)\left(\mathbf{V}_0^*(s)\times\mathbf{V}_0^*(s)\right)\\ &= \lambda^2(s).\end{aligned}$$

Hence,

$$d(\psi(s),\alpha(s^*)) = |\lambda(s)| = |r-s|.$$

Case II. This case can likewise be proved using the techniques of the proof of **Case I.** We need to take $\mathbf{V}_1^*(s)$ instead of $\mathbf{V}_0^*(s)$ in Eq. (3). Then, differentiating this new equation with respect to s, and by using Eq. (1), and take the octonionic inner product which obtained from the last equation with $\mathbf{V}_1^*(s)$, we have $\lambda'(s)=0$. From here, we get $\lambda(s)=r$. Finally, we obtain $d(\psi(s),\alpha(s^*)) = |\lambda(s)| = |r|$.

Case III. The proof is straightforward. ∎

Theorem 4. Let $\delta,\eta : I\subset\mathbb{R}\to O$ be octonionic curves with arc-length parameter s^*, and s, respectively.

Case I. If $\{\delta,\eta\}$ is octonionic involute-evolute curve couple in O, then we get $d(\eta(s),\delta(s^*)) = |r-s|$, where r is real number.

Case II. If $\{\eta,\delta\}$ is octonionic Bertrand curve couple in O, then we get $d(\eta(s),\delta(s^*)) = |r|$, where r is real number.

Case III. If $\{\eta,\delta\}$ is octonionic Mannheim curve couple in O, then we get $d(\eta(s),\delta(s^*)) = |r|$, where r is real number.

Proof. Case I. Let $\{\delta,\eta\}$ be octonionic involute-evolute curve couple in O. By the definition of octonionic involute-evolute curves, we have

$$\delta(s^*) = \eta(s) + \lambda(s)\mathbf{W}_0^*(s) \tag{5}$$

By differentiating the Eq. (5) with respect to s, and by using the Serret-Frenet equations given by Eq. (2), we get

$$\frac{d\delta(s^*)}{ds^*}\frac{ds^*}{ds} = \left(1+\lambda'(s)\right)\mathbf{W}_0^*(s) + \lambda(s)K^*(s)\mathbf{W}_1^*(s).$$

If we write $\frac{d\delta(s^*)}{ds^*} = \mathbf{W}_0(s^*)$, then we obtain

$$\mathbf{W}_0(s^*)\frac{ds^*}{ds} = \left(1+\lambda'(s)\right)\mathbf{W}_0^*(s) + \lambda(s)K^*(s)\mathbf{W}_1^*(s).$$

If we take the octonionic inner product of both sides of the last equation with $\mathbf{W}_0^*(s)$, then we can write

$$\begin{aligned}g\left(\mathbf{W}_0(s^*)\frac{ds^*}{ds},\mathbf{W}_0^*(s)\right) &= g(\left(1+\lambda'(s)\right))\mathbf{W}_0^*(s)\\ &\quad +\lambda(s)K^*(s)\mathbf{W}_1^*(s),\mathbf{W}_0^*(s))\\ &= \frac{1}{2}\{[\left(1+\lambda'(s)\right)\mathbf{W}_0^*(s)\\ &\quad +\lambda(s)K^*(s)\mathbf{W}_1^*(s)\times\overline{\mathbf{W}_0^*(s)}]\\ &\quad +[\mathbf{W}_0^*(s)\times\overline{(1+\lambda'(s))\mathbf{W}_0^*(s)}\\ &\quad \overline{+\lambda(s)K^*(s)\mathbf{W}_1^*(s)}]\},\end{aligned}$$

and thus

$$\begin{aligned}
g\left(\mathbf{W}_0(s^*)\frac{ds^*}{ds},\mathbf{W}_0^*(s)\right) &= \frac{1}{2}\left(1+\lambda'(s)\right)\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_0^*(s)}\right)\\
&\quad+\frac{1}{2}\lambda(s)K^*(s)\left(\mathbf{W}_1^*(s)\times\overline{\mathbf{W}_0^*(s)}\right)\\
&\quad+\frac{1}{2}\left(1+\lambda'(s)\right)\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_0^*(s)}\right)\\
&\quad+\frac{1}{2}\lambda(s)K^*(s)\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_1^*(s)}\right)\\
&= \left(1+\lambda'(s)\right)\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_0^*(s)}\right)\\
&\quad+\frac{1}{2}\lambda(s)K^*(s)[\left(\mathbf{W}_1^*(s)\times\overline{\mathbf{W}_0^*(s)}\right)\\
&\quad+\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_1^*(s)}\right)]\\
&= \left(1+\lambda'(s)\right)\|\mathbf{W}_0^*(s)\|^2\\
&\quad+\frac{1}{2}\lambda(s)K^*(s)[(\mathbf{V}_0^*(s)\times\mathbf{W}_0^*(s))\times\overline{\mathbf{W}_0^*(s)}\\
&\quad+\mathbf{W}_0^*(s)\times\overline{(\mathbf{V}_0^*(s)\times\mathbf{W}_0^*(s))}].\\
&= 1+\lambda'(s)+\frac{1}{2}\lambda(s)K^*(s)\{[(\mathbf{V}_0^*(s)\times\mathbf{W}_0^*(s))\\
&\quad\times\overline{\mathbf{W}_0^*(s)}]+[\mathbf{W}_0^*(s)\times\left(\overline{\mathbf{W}_0^*(s)}\times\overline{\mathbf{V}_0^*(s)}\right)]\}.
\end{aligned}$$

In the last equation, according to inverse properties and Moufang alternative properties, we get

$$(\mathbf{V}_0^*(s)\times\mathbf{W}_0^*(s))\times\overline{\mathbf{W}_0^*(s)}=\mathbf{V}_0^*(s),$$

and

$$\mathbf{W}_0^*(s)\times\left(\overline{\mathbf{W}_0^*(s)}\times\overline{\mathbf{V}_0^*(s)}\right)=\overline{\mathbf{V}_0^*(s)}.$$

So, we have

$$\begin{aligned}
g\left(\mathbf{W}_0(s^*)\frac{ds^*}{ds},\mathbf{W}_0^*(s)\right) &= 1+\lambda'(s)+\frac{1}{2}\lambda(s)K^*(s)\left(\mathbf{V}_0^*(s)+\overline{\mathbf{V}_0^*(s)}\right)\\
&= 0.
\end{aligned}$$

Since octonionic curve $\delta(s^*)$ is octonionic involute curve of octonionic curve $\eta(s)$, that is to say from the definition of octonionic involute-evolute curve couple, then we have

$$g\left(\mathbf{W}_0(s^*)\frac{ds^*}{ds},\mathbf{W}_0^*(s)\right)=\frac{ds^*}{ds}g(\mathbf{W}_0(s^*),\mathbf{W}_0^*(s))=0.$$

So, we get

$$1+\lambda'(s)=0. \tag{6}$$

By the last statements, we obtain

$$\lambda(s)=r-s.$$

By the Eqs. (5) and (6), we find

$$d(\eta(s),\delta(s^*))=\|\delta(s^*)-\eta(s)\|=\|\lambda(s)\mathbf{W}_0^*(s)\|.$$

On the other hand, we have

$$\begin{aligned} \left\|\lambda(s)\mathbf{T}_{\psi}(s)\right\|^{2} &= \lambda(s)\mathbf{W}_0^*(s)\times\overline{\lambda(s)\mathbf{W}_0^*(s)} \\ &= \lambda^2(s)\left(\mathbf{W}_0^*(s)\times\overline{\mathbf{W}_0^*(s)}\right) \\ &= \lambda^2(s)\left\langle\mathbf{W}_0^*(s),\mathbf{W}_0^*(s)\right\rangle \\ &= \lambda^2(s). \end{aligned}$$

Hence,

$$d(\eta(s),\delta(s^*)) = |\lambda(s)| = |r-s|.$$

Case II. The proof is straightforward.

Case III. The proof is straightforward. ■

Theorem 5. Let ψ,$\alpha : I\subset\mathbb{R}\to O_S$ be real spatial octonionic curves with arc-length parameter s, and s^*, respectively. If $\{\psi,\alpha\}$ is spatial octonionic Mannheim curve couple or spatial octonionic Bertrand curve couple in O_S, then the measure of the angle between the tangent vector fields of spatial special octonionic curves ψ, and α is constant.

Proof. Let ψ, $\alpha : I\subset\mathbb{R}\to O_S$ be real spatial octonionic Mannheim curve couple with arc-length parameter s, and s^*, respectively. Let's consider that

$$g(\mathbf{V}_0^*(s),\mathbf{V}_0(s^*)) = \cos\Theta,$$

where Θ is the angle between the octonion tangent Frenet Frame elements $\mathbf{V}_0$ and $\mathbf{V}_0^*$.

By differentiating the last statement with respect to s, and by using the Serret-Frenet equation given by Eq. (1), we get

$$\begin{aligned} \frac{d}{ds}g(\mathbf{V}_0^*(s),\mathbf{V}_0(s^*)) &= g\left(\frac{d\mathbf{V}_0^*(s)}{ds},\mathbf{V}_0(s^*)\right) \\ &\quad +g\left(\mathbf{V}_0^*(s),\frac{d\mathbf{V}_0(s^*)}{ds^*}\frac{ds^*}{ds}\right) \\ &= g(k_1^*(s)\mathbf{V}_1^*(s),\mathbf{V}_0(s^*)) \\ &\quad +g\left(\mathbf{V}_0^*(s),\frac{ds^*}{ds}k_1(s^*)\mathbf{V}_1(s^*)\right) \\ &= k_1^*(s)\,g(\mathbf{V}_1^*(s),\mathbf{V}_0(s^*)) \\ &\quad +k_1(s^*)\frac{ds^*}{ds}g(\mathbf{V}_0^*(s),\mathbf{V}_1(s^*)). \end{aligned}$$

First principal normal vector $\mathbf{V}_1^*(s)$ of ψ, and second principal normal vector of $\mathbf{V}_2(s^*)$ of α are linearly dependent according to definition of real spatial octonionic Mannheim curve. Since $\{\mathbf{V}_0,\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4,\mathbf{V}_5,\mathbf{V}_6\}$ is the Serret-Frenet frame, then $\mathbf{V}_0(s^*)$ is orthogonal to $\mathbf{V}_2(s^*)$. In this position, we have $g(\mathbf{V}_1^*(s),\mathbf{V}_0(s^*)) = 0$. Then, $\mathbf{V}_0^*(s)$ and $\mathbf{V}_1(s^*)$ are orthogonal to $\mathbf{V}_1^*(s)$ and $\mathbf{V}_2(s^*)$,respectively. Thus, we get $g(\mathbf{V}_0^*(s),\mathbf{V}_1(s^*)) = 0$. So,

$$\frac{d}{ds}g(\mathbf{V}_0^*(s),\mathbf{V}_0(s^*)) = 0,$$

and thus

$$g(\mathbf{V}_0^*(s),\mathbf{V}_0(s^*)) = constant.$$

The proof for spatial octonionic Bertrand curve couple in O_S can likewise be proved using the techniques of the proof for spatial octonionic Mannheim curve couple in O_S. ■

Theorem 6. Let $\eta,\ \delta : I \subset \mathbb{R} \to O$ be real octonionic curves with arc-length parameter s, and s^*, respectively. If $\{\eta,\ \delta\}$ is octonionic Mannheim curve couple or octonionic Bertrand curve couple in O, then the measure of the angle between the tangent vector fields of octonionic curves η and δ is constant.

Proof. Let $\eta,\ \delta : I \subset \mathbb{R} \to O$ be real octonionic Mannheim curves with arc-length parameter s, and s^*, respectively. Let's consider that

$$g\left(\mathbf{W}_0^*(s), \mathbf{W}_0(s^*)\right) = \cos\Omega.$$

By differentiating the last statement with respect to s, and by using the Serret-Frenet equation given by Eq. (2), we get

$$\begin{aligned}
\frac{d}{ds} g\left(\mathbf{W}_0^*(s), \mathbf{W}_0(s^*)\right) &= g\left(\frac{d\mathbf{W}_0^*(s)}{ds}, \mathbf{W}_0(s^*)\right) \\
&\quad + g\left(\mathbf{W}_0^*(s), \frac{d\mathbf{W}_0(s^*)}{ds^*}\frac{ds^*}{ds}\right) \\
&= g\left(K^*(s)\mathbf{W}_1^*(s), \mathbf{W}_0(s^*)\right) \\
&\quad + g\left(\mathbf{W}_0^*(s), \frac{ds^*}{ds}K(s^*)\mathbf{W}_1(s^*)\right) \\
&= K^*(s)\, g\left(\mathbf{W}_1^*(s), \mathbf{W}_0(s^*)\right) \\
&\quad + K(s^*)\frac{ds^*}{ds} g\left(\mathbf{W}_0^*(s), \mathbf{W}_1(s^*)\right).
\end{aligned}$$

First principal normal vector $\mathbf{W}_1^*(s)$ of η, and second principal normal vector of $\mathbf{W}_2(s^*)$ of δ are linearly dependent according to definition of real octonionic Mannheim curve couple. Additionally, $\mathbf{W}_0(s^*)$ is orthogonal to $\mathbf{W}_2(s^*)$, because $\{\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4, \mathbf{W}_5, \mathbf{W}_6,\ \mathbf{W}_7\}$ is the Serret-Frenet frame of δ. In this position, we have $g\left(\mathbf{W}_1^*(s), \mathbf{W}_0(s^*)\right) = 0$. Then, $\mathbf{W}_0^*(s)$ and $\mathbf{W}_1(s^*)$ are orthogonal to $\mathbf{W}_1^*(s)$ and $\mathbf{W}_2(s^*)$, sincerely. Thus, we get $g\left(\mathbf{W}_0^*(s), \mathbf{W}_1(s^*)\right) = 0$. So,

$$\frac{d}{ds} g\left(\mathbf{W}_0^*(s), \mathbf{W}_0(s^*)\right) = 0,$$

and thus

$$g\left(\mathbf{W}_0^*(s), \mathbf{W}_0(s^*)\right) = constant.$$

The proof for octonionic Bertrand curve couple in O can likewise be proved using the techniques of the proof for octonionic Mannheim curve couple in O. ■

Theorem 7. Let $\alpha : I \subset \mathbb{R} \to O_S$ be real spatial octonionic curves with arc-length parameter s. Then, α is spatial octonionic Mannheim curve if and only if

$$k_1^*(s) = \lambda(s)\left[\left(k_1^*(s)\right)^2 + \left(k_2^*(s)\right)^2\right],$$

where λ is constant distance between the corresponding points of Mannheim curves.

Proof. Let $\alpha : I \subset \mathbb{R} \to O_S$ be real spatial octonionic Mannheim curve, we can write

$$\alpha(s) = \psi(s) + \lambda(s)\mathbf{V}_1^*(s).$$

By differentiating the above equality, and by using the Serret-Frenet equations given by (1), we get

$$\frac{d\alpha(s)}{ds}=(\mathbf{1}-\lambda(s)k_1^*(s))\mathbf{V}_0^*(s)+\lambda'(s)\mathbf{V}_1^*(s)+\lambda(s)k_2^*(s)\mathbf{V}_2^*(s).$$

Since $\{\mathbf{V}_1^*(s),\mathbf{V}_2(s^*)\}$ is a linearly dependent set, then we have

$$\lambda'(s)=0.$$

This means that λ is constant. Thus we obtain

$$\frac{d\alpha(s)}{ds}=(\mathbf{1}-\lambda(s)k_1^*(s))\mathbf{V}_0^*(s)+\lambda(s)k_2^*(s)\mathbf{V}_2^*(s).$$

On the other hand, we find

$$\mathbf{V}_0(s)=\frac{d\alpha(s)}{ds}\frac{ds}{ds^*}=[(\mathbf{1}-\lambda(s)k_1^*(s))\mathbf{V}_0^*(s)+\lambda(s)k_2^*(s)\mathbf{V}_2^*(s)]\frac{ds}{ds^*}.$$

By taking the derivative of this equation with respect to s^*, and applying the Serret-Frenet formulas, then we get

$$\begin{aligned}\frac{d\mathbf{V}_0(s)}{ds}\frac{ds}{ds^*} &= [-\lambda(s)(k_1^*(s))'\mathbf{V}_0^*(s)+\\ &\left(k_1^*(s)-\lambda(s)(k_1^*(s))^2-\lambda(s)(k_2^*(s))^2\right)\mathbf{V}_1^*(s)\\ &+\lambda(s)(k_2^*(s))'\mathbf{V}_2^*(s)+\lambda(s)k_2^*(s)k_3^*(s)\mathbf{V}_3^*(s)]\frac{ds}{ds^*}\\ &+[(\mathbf{1}-\lambda(s)k_1^*(s))\mathbf{V}_0^*(s)+\lambda(s)k_2^*(s)\mathbf{V}_2^*(s)]\frac{d^2s}{ds^{*2}}.\end{aligned}$$

If we take the octonionic inner product of both sides of the last equation with $\mathbf{V}_1^*(s)$, getting

$$k_1^*(s)-\lambda(s)(k_1^*(s))^2-\lambda(s)(k_2^*(s))^2=0,$$

and thus

$$k_1^*(s)=\lambda(s)\left[(k_1^*(s))^2+(k_2^*(s))^2\right].$$

■

Theorem 8. Let $\delta: I\subset\mathbb{R}\to O$ be real octonionic curves with arc-length parameter s. Then, δ is octonionic Mannheim curve couple if and only if

$$K^*(s)=\lambda(s)\left[(K^*(s))^2+(k_1^*(s))^2\right],$$

where λ is constant distance between the corresponding points of Mannheim curves.

Proof. Let $\delta: I\subset\mathbb{R}\to O$ be real octonionic Mannheim curve couple, we can write

$$\delta(s)=\eta(s)+\lambda(s)\mathbf{W}_\mathbf{1}^*(s).$$

By differentiating the above equality and by using the Serret-Frenet equations, we get

$$\frac{d\delta(s)}{ds}=(\mathbf{1}-\lambda(s)K^*(s))\mathbf{W}_\mathbf{0}^*(s)+\lambda'(s)\mathbf{W}_\mathbf{1}^*(s)+\lambda(s)k_1^*(s)\mathbf{W}_\mathbf{2}^*(s).$$

Since $\left\{\mathbf{W}_\mathbf{1}^*(s),\mathbf{W}_\mathbf{2}(s^*)\right\}$ is a linearly dependent set, then we have

$$\lambda'(s)=0.$$

This means that λ is constant. Thus, we obtain

$$\frac{d\delta(s)}{ds} = (\mathbf{1} - \lambda(s) K^*(s)) \mathbf{W_0^*}(s) + \lambda(s) k_1^*(s) \mathbf{W_2^*}(s).$$

On the other hand, we find

$$\mathbf{W}_0(s) = \frac{d\delta(s)}{ds}\frac{ds}{ds^*} = [(\mathbf{1} - \lambda(s) K^*(s)) \mathbf{W_0^*}(s) + \lambda(s) k_1^*(s) \mathbf{W_2^*}(s)] \frac{ds}{ds^*}.$$

By taking the derivative of this equation with respect to s^*, and applying the Serret-Frenet formulas, we get

$$\begin{aligned}\frac{d\mathbf{W}_0(s)}{ds}\frac{ds}{ds^*} = & [-\lambda(s)(K^*(s))' \mathbf{W_0^*}(s) \\ & + \left(K^*(s) - \lambda(s)(K^*(s))^2 - \lambda(s)(k_1^*(s))^2\right) \mathbf{W_1^*}(s) \\ & + \lambda(s)(k_1^*(s))' \mathbf{W_2^*}(s) \\ & + \lambda(s) k_1^*(s)(k_2(s) - K(s))^*(s) \mathbf{W_3^*}(s)] \frac{ds}{ds^*} \\ & + [(\mathbf{1} - \lambda(s) K^*(s)) \mathbf{W_0^*}(s) + \lambda(s) k_1^*(s) \mathbf{W_2^*}(s)] \frac{d^2 s}{ds^{*2}}.\end{aligned}$$

If we take the octonionic inner product of both sides of the last equation with $\mathbf{W_1^*}(s)$, getting

$$K^*(s) - \lambda(s)(K^*(s))^2 - \lambda(s)(k_1^*(s))^2 = 0,$$

and thus

$$K^*(s) = \lambda(s)\left[(K^*(s))^2 + (k_1^*(s))^2\right].$$

■

Corollary 1. Let $\{\alpha, \psi\}$ and $\{\delta, \eta\}$ are spatial octonionic involute-evolute curve couple in O_S, and octonionic involute-evolute curve couple in O, respectively. For $\{\alpha, \psi\}$ is spatial octonionic involute-evolute curve couple in O_S, $\{\delta, \eta\}$ is octonionic involute-evolute curve couple in O if and only if

$$\frac{d}{ds} g(Im(\delta), Im(\eta)) + g\left(\frac{d}{ds}(Im(\delta)), \mathbf{V}_0^*\right) + g\left(\frac{d}{ds}(Im(\eta)), \mathbf{V}_0\right) = 0. \tag{7}$$

Proof. From the definition of the octonionic curve, we get the following curves

$$\alpha(s) = \sum_{i=1}^{7} \alpha_i(s^*) e_i,$$

$$\delta(s) = \sum_{i=0}^{7} \alpha_i(s^*) e_i,$$

$$\psi(s) = \sum_{i=1}^{7} \alpha_i^*(s) e_i,$$

and

$$\eta(s) = \sum_{i=0}^{7} \alpha_i^*(s) e_i.$$

Since $\{\alpha, \psi\}$ is spatial octonionic involute-evolute curve couple in O_S, $g(\mathbf{V}_0(s^*), \mathbf{V}_0^*(s)) = 0$. Let us compute the following inner product

$$\begin{aligned} & g(\mathbf{W}_0(s^*), \mathbf{W}_0^*(s)) \\ = & \; g\left(\alpha_0'(s^*) + \mathbf{V}_0(s^*), \alpha_0^{*\prime}(s) + \mathbf{V}_0^*(s)\right) \\ = & \; g\left(\alpha_0'(s^*), \alpha_0^{*\prime}(s)\right) + g\left(\alpha_0'(s^*), \mathbf{V}_0^*(s)\right) \\ & + g\left(\mathbf{V}_0(s^*), \alpha_0^{*\prime}(s)\right) + g(\mathbf{V}_0(s^*), \mathbf{V}_0^*(s)). \end{aligned} \tag{8}$$

Thus, we have $\frac{d}{ds} g(Im(\delta), Im(\eta)) + g\left(\frac{d}{ds}(Im(\delta)), \mathbf{V}_0^*\right) + g\left(\frac{d}{ds}(Im(\eta)), \mathbf{V}_0\right) = 0$.

Conversely, let Eq. (7) hold. Then $g(\mathbf{W}_0(s^*), \mathbf{W}_0^*(s)) = 0$. This proves the corollarly. ■

Theorem 9. Let $\eta : I \subset \mathbb{R} \to O$ be unit speed octonionic curve in Euclidean 8-space, and $\left\{\mathbf{W}_0^*, \mathbf{W}_1^*, \mathbf{W}_2^*, \mathbf{W}_3^*, \mathbf{W}_4^*, \mathbf{W}_5^*, \mathbf{W}_6^*, \mathbf{W}_7^*\right\}$ be the Serret-Frenet frame of η. The tangent of the octonionic $\mathbf{W}_0^*\mathbf{W}_1^* \cdots \mathbf{W}_7^*$-Smarandache curve is

$$\mathbf{W}_0(s^*) = \frac{\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^* + ik_{j-1}^* - mk_j^*\right](s)\,\mathbf{W}_j^*(s)}{\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^*(s) + ik_{j-1}^*(s) - mk_j^*(s)\right]^2}.$$

Proof. Octonionic $\mathbf{W}_0^*\mathbf{W}_1^* \cdots \mathbf{W}_7^*$-Smarandache curve is defined by

$$\Lambda(s^*) = \frac{1}{\sqrt{8}}\left(\sum_{i=1}^{7}\mathbf{W}_i^*(s)\right). \tag{9}$$

By differentiating (9), the following Eq. is obtained as follows

$$\frac{d\Lambda}{ds^*}\frac{(ds^*)}{ds} = \frac{1}{\sqrt{8}}\left(\sum_{i=1}^{7}(\mathbf{W}_i^*(s))'\right).$$

If we use the Serret-Frenet formula for USROC in Euclidean 8-space, then we get

$$\frac{d\Lambda}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{8}}\left(\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^* + ik_{j-1}^* - mk_j^*\right](s)\,\mathbf{W}_j^*(s)\right),$$

where

$$sgn\left(\mathbf{W}_j^*(s)\right) = \begin{cases} -1, & j = 0,3,5,6 \\ +1, & j = 1,2,4,7, \end{cases}$$

and

$$\begin{aligned} n &= 1,\ i = 0,\ m = 0\ for\ \ j = 0, \\ n &= 0,\ i = 0,\ m = 1\ for\ \ j = 1, \\ n &= 1,\ i = 1,\ m = 1\ for\ \ j = 2,3,4,5,6, \\ n &= 1,\ i = 1,\ m = 0\ for\ \ j = 7. \end{aligned}$$

On the other hand,

$$\mathbf{W}_0(s^*) = \frac{\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^* + ik_{j-1}^* - mk_j^*\right](s)\,\mathbf{W}_j^*(s)}{\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^*(s) + ik_{j-1}^*(s) - mk_j^*(s)\right]^2},$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{\sum_{j=0}^{7}\left[nsgn\left(\mathbf{W}_j^*(s)\right)K^*(s) + ik_{j-1}^*(s) - mk_j^*(s)\right]^2}{8}}.$$

■

4. Conclusion

Special curves are well known concepts in differential geometry. These curves are involute-evolute curve couple, Bertrand curve couple, Mannheim curve and Smarandache curve. These curves are defined as a special octonionic curves. Thus, the properties and characterizations for special curves in Euclidean space are investigated with the aid of the octonion algebra. As a result of this study, it was seen that special octonionic curves in octonionic space provide similar properties with specail curves in Euclidean space. Moreover, the relationship between the special octonionic curves and the special spatial octonionic curves are given.

References

[1] Aaron, D. W., ”The structure of E^6”, arXiv: 0711.3447v2, 2007.

[2] Ablamowicz, R., Lounesto, P., Parra, J. M., Clifford algebras with numeric and symbolic computations, *Birkhaerser,* Boston, 1996.

[3] Ali, A. T., ”Special smarandache curves in the Euclidean space”, *Int J of Math Comb.,* 2, 30-36, 2010.

[4] Baez, J. C., ”The Octonions”, *Bull Am Math Soc.,* 39, 145-205, 2001.

[5] Bektas, O., Yuce, S., ”Real variable Serret-Frenet formulae of an Octonion valued function (Octonionic curves)”, *Colloquium on Combinatorics,* 3, 2014.

[6] Bektas, O., Yuce, S., ”Serret Frenet formulas for octonionic curves”, *Boletim da Sociedade Paranaense de Matematica,* doi:10.5269/bspm.v38i3.34780(In press).

[7] Bertrand, M., ”M’emoire sur la th’eorie des courbes’a double courbure”, *ComptesRendus,* 1850.

[8] Bharathi, K., Nagaraj, M., ”Quaternion valued function of a real variable Serret-Frenet formulae”, *Indian J Pure Appl Math.,* 18, 507-511, 1987.

[9] Bhupendra, C. C., Chauhan, N. O., ”Octonion formulation of seven dimensional vector space”, *Fundamental J Math Phys.,* 1, 41-52, 2011.

[10] Bioche, Ch., ”Sur les courbes de M. Bertrand”, *Bull Soc Math France*, 17, 109-112, 1889.

[11] Burke, J. K., ”Bertrand curves sssociated with a pair of Curves”, *Math Magazine,* 34, 60-62, 1960.

[12] Cayley, A., ”On Jacobi’s elliptic functions, in reply to the Rev. B. Bronwin and on quaternions”, *Philos. Mag.,* 26, 208-211, 1845.

[13] Cetin, M., Kocayigit, H., ”On the Quaternionic smarandache curves in Euclidean 3-Space”, *Int J Contemp Math Sci.,* 8, 139-150, 2003.

[14] Corinne, A.M., Tevian, D., ”Octonions, E^6, and Particle Physics”, arXiv:0911.2253v2, 2010.

[15] Ekmekci, K., Ilarslan, K., ”On Bertrand curves and their characterization”, *Differ Geom Dyn Syst.,* 3, 17-24, 2001.

[16] Fenn, R., Geometry, Undergraduate Mathematics Series *Springer*, 2007.

[17] Gentili, G., Stoppato, C., Struppa, D. C., Vlacci, F., ”Recent developments for regular functions of a hypercomplex variable”, Conference on quaternionic and Clifford analysis proceedings ed., *Birkauser*, 168, 2008.

[18] Gluck, H., ”Higher curvatures of curves in Euclidean Space”, *The Am Math Monthly,* 73, 699-704, 1966.
[19] Gok, I., Kahraman, F., ”Characterizations of the Quaternionic Bertrand curve in Euclidean Space E^4”, arXiv:1302.2809, 2015.
[20] Graves, R. P., Life of Sir William Rowan Hamilton, *Arno Press*, New York, 1975.
[21] Hamilton, W. R., Elements of quaternions, *Chelsea Publications,* New York, 1969.
[22] Kecilioglu, O., Ilarslan, K., ”Quaternionic Bertrand curves in Euclidean 4-Space”, *Bull of Math Analysis Appl,* 5, 27-38, 2013.
[23] Liu, H., Wang, F., ”Mannheim partner curves in 3-space”, *J Geom.,* 88, 120-126, 2008.
[24] Lounesto, P., Clifford Algebras and Spinors, *Cambridge University Press,* Cambridge, 1997.
[25] Lounesto, P., ”Octonions and Triality”, *Advances in Applied Clifford Algebras*, 11, 191-213, 2001.
[26] Okuyucu, O.Z., ”Characterizations of the Quaternionic Mannheim curves in Euclidean space E^4”, *Int J of Math Comb.,* 2, 2013.
[27] Ozyilmaz, E., Yilmaz, S., ”Involute-evolute Curve couples in the Euclidean 4-Space”, *Int J Open Problems Compt Math*, 2, 168-174, 2009.
[28] Sabinin, L., Sbitneva, L., Shestakov, I. P., Non-associative algebra and its applications, *CRC Press*, 235, 2006.
[29] Schray, J., Manogue, C. A., ”Foundations of Physics”, *Springer* 26(1), 17-70, 1996.
[30] Soyfidan, T., ”Quaternionic Involute-evolute curve couples”, Master Thesis, Sakarya University, Turkey, 2011.
[31] Turgut, A., Esin, E., ”Involute-evolute Curve Couples of Higher Order in and Their Horizontal Lifts in $\mathbb{R}^n$”, *Comm Fac Sci Univ Ankara Ser. A 1 Math. Statist*, 41, 125-130, 1992.
[32] Urhammer, E., Real Division Algebras, http://www.math.ku.dk/˜moller/
undervisning/aktuel/rap2/emil2.pdf).
[33] Ward, J. P., Quaternions and Cayley Numbers Algebra and Applications, *Kluwer Academic Publishers*, London, 1997.